ZETA FUNCTIONS OF REGULAR ARITHMETIC SCHEMES AT $s = 0$

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Abstract. Lichtenbaum conjectured in [22] the existence of a Weil-étale cohomology in order to describe the vanishing order and the special value of the Zeta function of an arithmetic scheme $\mathcal{X}$ at $s = 0$ in terms of Euler-Poincaré characteristics. Assuming the (conjectured) finite generation of some motivic cohomology groups we construct such a cohomology theory for regular schemes proper over Spec($\mathbb{Z}$). In particular, we compute (unconditionally) the right Weil-étale cohomology of number rings and projective spaces over number rings. We state a precise version of Lichtenbaum's conjecture, which expresses the vanishing order (resp. the special value) of the Zeta function $\zeta(\mathcal{X}, s)$ at $s = 0$ as the rank (resp. the determinant) of a single perfect complex of abelian groups $R\Gamma_{W,c}(\mathcal{X}, \mathbb{Z})$. Then we relate this conjecture to Soulé's conjecture and to the Tamagawa Number Conjecture. Lichtenbaum's conjecture for projective spaces over the ring of integers of an abelian number field follows.

1. Introduction

In [22] Lichtenbaum conjectured the existence of a Weil-étale cohomology in order to describe the vanishing order and the special value of the Zeta function of an arithmetic scheme i.e. a separated scheme of finite type over $\mathbb{Z}$ at a negative integer $s = n$ in terms of Euler-Poincaré characteristics. More precisely, we have the following

Conjecture 1.1. *(Lichtenbaum) Let $\mathcal{X}$ be an arithmetic scheme. There exist cohomology groups $H^i_{W,c}(\mathcal{X}, \mathbb{Z})$ and $H^i_{W,c}(\mathcal{X}, \mathbb{R})$ such that the following holds.*

(1) *The groups $H^i_{W,c}(\mathcal{X}, \mathbb{Z})$ are finitely generated and zero for i large.*
(2) *The natural map from $\mathbb{Z}$ to $\mathbb{R}$-coefficients induces isomorphisms*
$$H^i_{W,c}(\mathcal{X}, \mathbb{Z}) \otimes \mathbb{R} \simeq H^i_{W,c}(\mathcal{X}, \mathbb{R}).$$
(3) *There exists a canonical class $\theta \in H^1_W(\overline{\mathcal{X}}, \mathbb{R})$ such that cup-product with θ turns the sequence*
$$... \xrightarrow{\cup\theta} H^i_{W,c}(\mathcal{X}, \tilde{\mathbb{R}}) \xrightarrow{\cup\theta} H^{i+1}_{W,c}(\mathcal{X}, \tilde{\mathbb{R}}) \xrightarrow{\cup\theta} ...$$
into a bounded acyclic complex of finite dimensional vector spaces.

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(4) *The vanishing order of the zeta function* $\zeta(\mathcal{X},s)$ *at* $s=0$ *is given by the formula*

$$\mathrm{ord}_{s=0}\zeta(\mathcal{X},s)=\mathrm{rank}_{\mathbb{Z}}\mathrm{R}\Gamma_{W,c}(\mathcal{X},\mathbb{Z}):=\sum_{i\geq 0}(-1)^i.i.\mathrm{rank}_{\mathbb{Z}}H^i_{W,c}(\mathcal{X},\mathbb{Z})$$

(5) *The leading coefficient in the Taylor development of* $\zeta(\mathcal{X},s)$ *at* $s=0$ *is given up to sign by*

$$\mathbb{Z}\cdot\lambda(\zeta^*(\mathcal{X},0))=\mathrm{det}_{\mathbb{Z}}\mathrm{R}\Gamma_{W,c}(\mathcal{X},\mathbb{Z}).$$

where $\lambda:\mathbb{R}\xrightarrow{\sim}\mathrm{det}_{\mathbb{R}}\mathrm{R}\Gamma_{W,c}(\mathcal{X},\mathbb{Z})_{\mathbb{R}}$ *is induced by* (3).

Lichtenbaum defined such cohomology groups for smooth varieties over finite fields in [21], but similar attempts to define such a cohomology for flat arithmetic schemes failed. In [22] Lichtenbaum gave the first construction for number rings. He defined a Weil-étale topology which bears the same relation to the usual étale topology as the Weil group does to the Galois group. Under a vanishing statement, he was able to show that his cohomology miraculously yields the value of Dedekind zeta functions at $s=0$. But this cohomology with coefficients in $\mathbb{Z}$ was then shown to be infinitely generated in even degrees $i\geq 4$ [6]. Consequently, Lichtenbaum's complex computing the $\mathbb{Z}$-cohomology needs to be artificially truncated in the case of number rings, and is not helpful for flat schemes of dimension greater than 1. Moreover, it seems impossible to construct relevant sheaves $\mathbb{Z}(n)$ for $n\neq 0$ on the Weil-étale topology. However, Lichtenbaum's definition does work with $\mathbb{R}$-coefficients, and this fact extends to higher dimensional arithmetic schemes [7]. Moreover, the geometric intuition given by Lichtenbaum's Weil-étale topos provides a guiding light for any other definition, as it is the case in this paper.

Here we propose an alternative approach, whose basic idea can be explained as follows. The Weil group is (non-canonically) defined as an extension of the Galois group by the idèle class group "corresponding" to the fundamental class of class field theory. In this paper we use étale duality for arithmetic schemes rather than class field theory, in order to obtain a (canonical) extension of the étale $\mathbb{Z}$-cohomology by the dual of motivic $\mathbb{Q}(d)$-cohomology. Here d is the dimension of the scheme we consider. This idea was suggested by [5], [9] and [24].

This method allows us to define the Weil-étale cohomology groups of an arithmetic scheme $\mathcal{X}$ satisfying the following conjecture. We fix a regular scheme $\mathcal{X}$ proper over $\mathbb{Z}$ of Krull dimension d, and we denote by $\mathbb{Z}(d)$ Bloch's cycle complex.

Conjecture 1.2. *(Lichtenbaum) The étale motivic cohomology groups* $H^i(\mathcal{X}_{et},\mathbb{Z}(d))$ *are finitely generated for* $0\leq i\leq 2d$.

This conjecture holds for projective spaces over number rings. If $\mathcal{X}$ is defined over a finite field $\mathbb{F}_q$, a strong version of Conjecture 1.2 is equivalent to a previous conjecture of Lichtenbaum denoted by $L(\mathcal{X},d)$ in [9]. Conjecture $L(\mathcal{X},d)$ is known for the class $A(\mathbb{F}_q)$ of projective smooth varieties over $\mathbb{F}_q$ which can be constructed out of products of smooth projective curves by union, base extension

and blow ups (this includes abelian varities, unirational varieties of dimension at most 3 and Fermat hypersurfaces).

In [7] the Weil-étale topos $\mathcal{X}_{\infty,W}$ is naturally associated to the trivial action of the topological group $\mathbb{R}$ on the topological space $\mathcal{X}_\infty := \mathcal{X}(\mathbb{C})/G_\mathbb{R}$, where $\mathcal{X}(\mathbb{C})$ is given with the complex topology. We set $\overline{\mathcal{X}} := (\mathcal{X}, \mathcal{X}_\infty)$. Our first main result is the following

Theorem 1.3. *If $\mathcal{X}$ satisfies Conjecture 1.2 then there exist complexes $\mathrm{R}\Gamma_W(\mathcal{X},\mathbb{Z})$ and $\mathrm{R}\Gamma_{W,c}(\mathcal{X},\mathbb{Z})$ such that the following properties hold.*

- *The complex $\mathrm{R}\Gamma_W(\overline{\mathcal{X}},\mathbb{Z})$ is contravariantly functorial.*
- *There is a canonical morphism $\mathrm{R}\Gamma_W(\mathcal{X},\mathbb{Z}) \to \mathrm{R}\Gamma(\mathcal{X}_{\infty,W},\mathbb{Z})$ (see Proposition 9.2 for a precise statement) and an exact triangle*
$$\mathrm{R}\Gamma_{W,c}(\mathcal{X},\mathbb{Z}) \to \mathrm{R}\Gamma_W(\mathcal{X},\mathbb{Z}) \to \mathrm{R}\Gamma(\mathcal{X}_{\infty,W},\mathbb{Z}) \to \mathrm{R}\Gamma_{W,c}(\mathcal{X},\mathbb{Z})[1]$$
- *The groups $H^i_W(\mathcal{X},\mathbb{Z})$ and $H^i_{W,c}(\mathcal{X},\mathbb{Z})$ are finitely generated and zero for i large.*
- *The Weil-étale cohomology groups $H^i_W(\overline{\mathcal{X}},\mathbb{Z})$ form an integral model for l-adic cohomology: for any prime number l and any $i \in \mathbb{Z}$ there is a canonical isomorphism*
$$H^i_W(\overline{\mathcal{X}},\mathbb{Z}) \otimes \mathbb{Z}_l \simeq H^i(\overline{\mathcal{X}}_{et},\mathbb{Z}_l)$$
- *If $\mathcal{X}$ has characteristic p then there is a canonical isomorphism*
$$\mathrm{R}\Gamma(\mathcal{X}_W,\mathbb{Z}) \xrightarrow{\sim} \mathrm{R}\Gamma_W(\mathcal{X},\mathbb{Z})$$
where the left hand side is the cohomology of the Weil-étale topos [21] and the right hand side is the complex defined in this paper. Moreover, Conjecture 1.1 holds for $\mathcal{X}$.
- *If $\mathcal{X} = \mathrm{Spec}(\mathcal{O}_F)$ is the spectrum of a (totally imaginary) number ring, then $\mathcal{X}$ satisfies Conjecture 1.2 and there is a canonical isomorphism*
$$\tau_{\leq 3}\mathrm{R}\Gamma(\mathcal{X}_W,\mathbb{Z}) \xrightarrow{\sim} \mathrm{R}\Gamma_W(\mathcal{X},\mathbb{Z})$$
where $\tau_{\leq 3}\mathrm{R}\Gamma(\mathcal{X}_W,\mathbb{Z})$ is the truncation of Lichtenbaum's complex [22] and the right hand side is the complex defined in this paper. Moreover, Conjecture 1.1 holds for $\mathcal{X}$.

Notice that the same formalism is used to treat flat arithmetic schemes and schemes over finite fields (see [22] Question 1 in the Introduction). To go further we need to assume the following conjecture, which is a special case of a natural refinement for arithmetic schemes of the classical conjecture of Beilinson relating motivic cohomology to Deligne cohomology. Let $\mathcal{X}$ be a regular, proper and flat arithmetic scheme. Assume that $\mathcal{X}$ is connected of dimension d.

Conjecture 1.4. *(Beilinson-Flach) The Beilinson regulator*
$$H^{2d-1-i}(\mathcal{X},\mathbb{Q}(d))_\mathbb{R} \to H^{2d-1-i}_\mathcal{D}(\mathcal{X}_{/\mathbb{R}},\mathbb{R}(d))$$
is an isomorphism for $1 \leq i \leq 2d-1$ and there is an exact sequence
$$0 \to H^{2d-1}(\mathcal{X},\mathbb{Q}(d))_\mathbb{R} \to H^{2d-1}_\mathcal{D}(\mathcal{X}_{/\mathbb{R}},\mathbb{R}(d)) \to CH^0(\mathcal{X}_\mathbb{Q})^*_\mathbb{R} \to 0$$

Now we can state our second main result.

Theorem 1.5. *Assume that $\mathcal{X}$ satisfies Conjectures 1.2 and 1.4.*

- *Then Statements* (1), (2) *and* (3) *of Conjecture 1.1 hold.*
- *Statement* (4) *of Conjecture 1.1, i.e. the identity*
$$\mathrm{ord}_{s=0}\zeta(\mathcal{X},s) = \mathrm{rank}_{\mathbb{Z}}\mathrm{R}\Gamma_{W,c}(\mathcal{X},\mathbb{Z})$$
is equivalent to Soulé's Conjecture [26] *for the vanishing order of $\zeta(\mathcal{X},s)$ at $s=0$.*
- *(Flach-Morin) If the L-functions of the motives $h^i(\mathcal{X}_{\mathbb{Q}})$ satisfy the expected functional equation then Statement* (4) *holds.*
- *(Flach-Morin) Assume that $\mathcal{X}$ is smooth proper over a number ring. Then Statement* (5) *of Conjecture 1.1, i.e. the identity*
$$\mathbb{Z}\cdot\lambda(\zeta^*(\mathcal{X},0)) = \det_{\mathbb{Z}}\mathrm{R}\Gamma_{W,c}(\mathcal{X},\mathbb{Z})$$
is equivalent to the Tamagawa Number Conjecture for the motive $h(\mathcal{X}_{\mathbb{Q}})$.

This result shows that the Weil-étale point of view is compatible with the Bloch-Kato Conjecture, answering a question of Lichtenbaum (see [22] Question 2 in the Introduction). We obtain the first cases where Conjecture 1.1 is known for flat arithmetic schemes:

Corollary 1.6. *Conjecture 1.1 holds in the following cases:*

- *For $\mathcal{X} = \mathrm{Spec}(\mathcal{O}_F)$ the spectrum of a number ring.*
- *For $\mathcal{X} = \mathbb{P}^n_{\mathcal{O}_F}$ the projective n-space over the ring of integers in an abelian number field $F/\mathbb{Q}$.*
- *For $\mathcal{X} = \mathbb{P}^n_{\mathcal{O}_F} - Y$ where $F/\mathbb{Q}$ is abelian and $Y = \coprod_{i=1}^{i=s} Y_i$ is the disjoint union in $\mathbb{P}^n_{\mathcal{O}_F}$ of varieties Y_i over finite fields $\mathbb{F}_{q_i}$ lying in $A(\mathbb{F}_{q_i})$.*

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Contents

2. The Artin-Verdier étale topology

2.1. **The Artin-Verdier étale topos.** Let $\mathcal{X}$ be a proper, regular and connected arithmetic scheme of dimension $d := \dim(\mathcal{X})$. Let

$$\mathcal{X}_\infty := \mathcal{X}(\mathbb{C})/G_{\mathbb{R}}$$

be the quotient topological space of $\mathcal{X}(\mathbb{C})$ by $G_{\mathbb{R}}$, where $\mathcal{X}(\mathbb{C})$ is given with the complex topology. We consider the Artin-Verdier étale topos $\overline{\mathcal{X}}_{et}$ given with the open-closed decomposition of topoi

$$\varphi : \mathcal{X}_{et} \to \overline{\mathcal{X}}_{et} \leftarrow Sh(\mathcal{X}_\infty) : u_\infty$$

where $\mathcal{X}_{et}$ is the usual étale topos of the scheme $\mathcal{X}$ (i.e. the category of sheaves of sets on the small étale site of $\mathcal{X}$) and $Sh(\mathcal{X}_\infty)$ is the category of sheaves of sets on the topological space $\mathcal{X}_\infty$ (see [7] Section 4). For any abelian sheaf A on $\mathcal{X}_{et}$ and any $n > 0$ the sheaf $R^n\varphi_*A$ is is 2-torsion and concentrated on $\mathcal{X}(\mathbb{R})$. In particular, if $\mathcal{X}(\mathbb{R}) = \emptyset$ then

$$\phi_* \simeq R\phi_*$$

For $\mathcal{X}(\mathbb{C}) = \emptyset$ one has $\mathcal{X}_{et} = \overline{\mathcal{X}}_{et}$.

2.2. **Bloch's cycle complex.** Let $\mathbb{Z}(n) := z^n(-, 2n - *)$ be Bloch's complex (see [8], [10], [20]), which we consider as a complex of abelian sheaves on the small étale site of $\mathcal{X}$. We denote by $H^i(\mathcal{X}, \mathbb{Q}(n)) := H^i(\mathcal{X}_{Zar}, \mathbb{Q}(n))$ the Zariski hypercohomology of the cycle complex. For $\mathcal{X}(\mathbb{R}) = \emptyset$, we still denote by $\mathbb{Z}(n) = \varphi_*\mathbb{Z}(n)$ and $\mathbb{Q}(n) = \varphi_*\mathbb{Q}(n)$ the push-forward of the cycle complex on $\overline{\mathcal{X}}_{et}$, and by $H^i(\overline{\mathcal{X}}_{et}, \mathbb{Z}(n))$ the étale hypercohomology of $\mathbb{Z}(n)$.

3. The conjectures $\mathbf{L}(\overline{\mathcal{X}}_{et}, d)$ and $\mathbf{L}(\overline{\mathcal{X}}_{et}, d)_{\geq 0}$

The following conjecture is an étale version of (a special case of) the motivic Bass Conjecture (see [15] Conjecture 37). The kernel and the cokernel of the restriction map $H^i(\overline{\mathcal{X}}_{et}, \mathbb{Z}(d)) \to H^i(\mathcal{X}_{et}, \mathbb{Z}(d))$ are both finite groups killed by 2. Here the complex $\mathbb{Z}(d)$ on $\overline{\mathcal{X}}$ denotes the dualizing complex for the Artin-Verdier étale cohomology.

Conjecture 3.1. $\mathbf{L}(\overline{\mathcal{X}}_{et}, d)$ *Let $\mathcal{X}$ be a proper, regular and connected arithmetic scheme of dimension d. Then the group $H^i(\overline{\mathcal{X}}_{et}, \mathbb{Z}(d))$, or equivalently $H^i(\mathcal{X}_{et}, \mathbb{Z}(d))$, is finitely generated for any $i \leq 2d$.*

In order to define the Weil-étale cohomology we shall assume a weak version of the previous conjecture.

Conjecture 3.2. $\mathbf{L}(\overline{\mathcal{X}}_{et}, d)_{\geq 0}$ *Let $\mathcal{X}$ be a proper, regular and connected arithmetic scheme of dimension d. Then the group $H^i(\overline{\mathcal{X}}_{et}, \mathbb{Z}(d))$, or equivalently $H^i(\mathcal{X}_{et}, \mathbb{Z}(d))$, is finitely generated for $0 \leq i \leq 2d$.*

The following is a special case of the Beilinson-Soulé vanishing conjecture.

Conjecture 3.3. *(Beilinson-Soulé) For any $i < 0$ we have $H^i(\mathcal{X}, \mathbb{Q}(d)) = 0$.*

Lemma 4.1 shows that $H^i(\mathcal{X}_{et}, \mathbb{Z}(d)) = H^i(\mathcal{X}, \mathbb{Q}(d))$ for any $i < 0$, hence we have the

Proposition 3.4. *A scheme $\mathcal{X}$ as above satisfies Conjecture 3.1 if and only if it satisfies both Conjecture 3.2 and Conjecture 3.3*

3.1. Projective spaces over number rings.

Proposition 3.5. *Any number ring $\mathcal{X} = \mathrm{Spec}(\mathcal{O}_F)$ satisfies Conjecture 3.1.*

Proof. In view of the quasi-isomorphism of complexes of étale sheaves

$$\mathbb{Z}(1) \simeq \mathbb{G}_m[-1]$$

the result follows from the finite generation of the unit group and finiteness of the class group of a number field. □

Proposition 3.6. *The projective space $\mathbb{P}^n_{\mathcal{O}_F}$ over a number ring satisfies Conjecture 3.1.*

Proof. This is proven in Theorem 15.7 (1). □

3.2. **Varieties over finite fields.** Let $Y = \mathcal{X}$ be a smooth projective scheme over $k = \mathbb{F}_q$ of dimension $d = \dim(Y)$. The Weil-étale topos Y_W (see [21] and [9]) gives rise to the right arithmetic cohomology $H^i(Y_W, \mathbb{Z}(n))$. The following conjecture is due to Lichtenbaum.

Conjecture 3.7. $\mathbf{L}(Y_W, n)$ *The Weil-étale cohomology group $H^i(Y_W, \mathbb{Z}(n))$ is finitely generated for every i.*

The following conjecture is due to B. Kahn (see [16] and [9]).

Conjecture 3.8. $\mathbf{K}(Y_W,n)$ *For every prime* l*, and any* i*, the natural map induces an isomorphism*

$$H^i(Y_W,\mathbb{Z}(n))\otimes\mathbb{Z}_l\simeq H^i_{cont}(Y,\mathbb{Z}_l(n))$$

Consider the full subcategory $A(k)$ of the category of smooth projective varieties over the finite field k generated by product of curves and the following operations:

(1) If X and Y are in $A(k)$ then $X\coprod Y$ is in $A(k)$.

(2) If Y is in $A(k)$ and there are morphisms $c:X\to Y$ and $c':Y\to X$ in the category of Chow motives, such that $c'\circ c:X\to X$ is multiplication by a constant, then X is in $A(k)$.

(3) If k'/k is a finite extension and $X\times_k k'$ is in $A(k)$, then X is in $A(k)$.

(4) If Y is a closed subscheme of X with X and Y in $A(k)$, then the blow-up X' of X is in $A(k)$.

The following result is due to T. Geisser [9].

Theorem 3.9. *(Geisser) Let* Y *be a smooth projective variety of dimension* d*.*

- One has $\mathbf{K}(Y_W,n)+\mathbf{K}(Y_W,d-n)\Rightarrow\mathbf{L}(Y_W,n)\Rightarrow\mathbf{K}(Y_W,n)$*.*

- Moreover, if Y *belongs to* $A(k)$ *then* $\mathbf{L}(Y_W,n)$ *holds for* $n\leq 1$ *and* $n\geq d-1$*.*

Proposition 3.10. *Let* Y *be a connected smooth projective scheme over a finite field of dimension* d*. Then we have*

$$\mathbf{L}(Y_W,d)\Leftrightarrow\mathbf{L}(Y_{et},d)$$

Proof. By [9], one has an exact sequence

(1)

$$...\to H^i(Y_{et},\mathbb{Z}(n))\to H^i(Y_W,\mathbb{Z}(n))\to H^{i-1}(Y_{et},\mathbb{Q}(n))\to H^{i+1}(Y_{et},\mathbb{Z}(n))\to ...$$

With rational coefficients, this exact sequence yields isomorphisms

(2) $\quad H^i(Y_W,\mathbb{Z}(n))\otimes\mathbb{Q}\simeq H^i(Y_W,\mathbb{Q}(n))\simeq H^i(Y_{et},\mathbb{Q}(n))\oplus H^{i-1}(Y_{et},\mathbb{Q}(n)).$

For $n=d$ the following is known (see [16] proof of Corollaire 3.10).

- One has $H^i(Y_{et},\mathbb{Z}(d))=0$ for $i>2d+2$.
- There is an isomorphism $H^{2d+2}(Y_{et},\mathbb{Z}(d))\simeq\mathbb{Q}/\mathbb{Z}$.
- The group $H^{2d}(Y_{et},\mathbb{Z}(d))\simeq CH^d(Y)$ is finitely generated of rank 1.

Assume now that Conjecture $\mathbf{L}(Y_W,d)$ holds. Let us first show that $H^i(Y_W,\mathbb{Z}(d))$ is finite for $i\neq 2d,2d+1$. By Theorem 3.9, Conjecture $\mathbf{K}(Y_W,d)$ holds, i.e. we have an isomorphism

$$H^i(Y_W,\mathbb{Z}(d))\otimes\mathbb{Z}_l\simeq H^i_{cont}(Y,\mathbb{Z}_l(d))$$

for any l and any i. But for $i\neq 2d,2d+1$, it is known that $H^i_{cont}(Y,\mathbb{Z}_l(d))$ is finite for any l and zero for almost all l (see [16] Proof of Corollaire 3.8 for references). Hence $H^i(Y_W,\mathbb{Z}(d))$ is finite for $i\neq 2d,2d+1$. Then (2) gives $H^i(Y_{et},\mathbb{Q}(d))=0$ for $i<2d$, hence $H^i(Y_{et},\mathbb{Q}(d))=0$ for $i\neq 2d$. The exact sequence (1) then shows that $H^i(Y_{et},\mathbb{Z}(d))\to H^i(Y_W,\mathbb{Z}(d))$ is injective for $i\leq 2d+1$, hence that $H^i(Y_{et},\mathbb{Z}(d))$ is finitely generated for $i\leq 2d+1$.

Assume now that Conjecture $\mathbf{L}(Y_{et},d)$ holds. By Proposition 4.3 the natural map

$$H^i(Y_{et},\mathbb{Z})\to\mathrm{Hom}(H^{2d+2-i}(Y_{et},\mathbb{Z}(d)),\mathbb{Q}/\mathbb{Z})$$

is an isomorphism of abelian groups for $i \geq 1$. But $H^i(Y_{et}, \mathbb{Z})$ is known to be finite for $i \neq 0, 2$ and zero for $i \geq 2d+2$ (except for $d = 0$). It follows that $H^j(Y_{et}, \mathbb{Z}(d))$ is finite for $j \neq 2d, 2d+2$. We get $H^i(Y_{et}, \mathbb{Q}(d)) = 0$ for $i \neq 2d$. Moreover we have $H^{2d}(Y_{et}, \mathbb{Q}(d)) = CH^d(Y)_{\mathbb{Q}} \simeq \mathbb{Q}$. Finally, one has $H^{2d+1}(Y_{et}, \mathbb{Z}(d)) = 0$ since $H^1(Y_{et}, \mathbb{Z}) = 0$ and $H^{2d}(Y_{et}, \mathbb{Z}(d))$ is finitely generated as mentioned above. Hence $\mathbf{L}(Y_W, d)$ follows from (1) and (2) since the map

$$\mathbb{Q} \simeq H^{2d}(Y_{et}, \mathbb{Q}(d)) \to H^{2d+2}(Y_{et}, \mathbb{Z}(d)) \simeq \mathbb{Q}/\mathbb{Z}$$

in the sequence (1) is the obvious surjection. □

Corollary 3.11. *Any variety Y in $A(k)$ satisfies $\mathbf{L}(Y_{et}, d)$.*

Proof. This follows from Theorem 3.9 and Proposition 3.10. □

4. The morphism $\alpha_{\mathcal{X}}$

Let $\mathcal{X}$ be a proper regular connected arithmetic scheme of dimension d. The purpose of this section is to show Theorem 4.4. We need a version of duality for arithmetic schemes with $\mathbb{Z}$-coefficients which requires Conjecture 3.2.

4.1. **The case $\mathcal{X}(\mathbb{R}) = \emptyset$.** Recall that in this case we have $\phi_* = R\phi_*$.

Lemma 4.1. *Let $\mathcal{X}$ be a proper, regular and connected arithmetic scheme of dimension d such that $\mathcal{X}(\mathbb{R})$ is empty. We have*

$$\begin{aligned} H^i(\overline{\mathcal{X}}_{et}, \mathbb{Z}(d)) &= 0 \text{ for } i > 2d+2 \\ &\simeq \mathbb{Q}/\mathbb{Z} \text{ for } i = 2d+2 \\ &= 0 \text{ for } i = 2d+1 \\ &\simeq H^i(\mathcal{X}, \mathbb{Q}(d)) \text{ for } i < 0 \end{aligned}$$

Proof. We have $H^i(\overline{\mathcal{X}}_{et}, \mathbb{Q}(d)) \simeq H^i(\mathcal{X}, \mathbb{Q}(d))$ for any i (see [8] Proposition 3.6) and $H^i(\mathcal{X}, \mathbb{Q}(d)) = 0$ for any $i > 2d$. The exact triangle

$$\mathbb{Z}(d) \to \mathbb{Q}(d) \to \mathbb{Q}/\mathbb{Z}(d) \tag{3}$$

gives

$$H^i(\overline{\mathcal{X}}_{et}, \mathbb{Z}(d)) \simeq H^{i-1}(\overline{\mathcal{X}}_{et}, \mathbb{Q}/\mathbb{Z}(d)) \simeq \varinjlim H^{i-1}(\overline{\mathcal{X}}_{et}, \mathbb{Z}/n\mathbb{Z}(d))$$

for $i \geq 2d+2$. The result for $i \geq 2d+2$ follows from

$$H^{i-1}(\overline{\mathcal{X}}_{et}, \mathbb{Z}/n\mathbb{Z}(d)) = \mathrm{Ext}^{i-1}_{\overline{\mathcal{X}}, \mathbb{Z}/n\mathbb{Z}}(\mathbb{Z}/n\mathbb{Z}, \mathbb{Z}/n\mathbb{Z}(d)) \tag{4}$$

$$\simeq \mathrm{Ext}^{i}_{\overline{\mathcal{X}}}(\mathbb{Z}/n\mathbb{Z}, \mathbb{Z}(d)) \tag{5}$$

$$\simeq H^{2d+2-i}(\overline{\mathcal{X}}_{et}, \mathbb{Z}/n\mathbb{Z})^D \tag{6}$$

where (5) and (6) are given by [11] Lemma 2.4 and [11] Theorem 7.8 respectively, and $H^{2d+2-i}(\overline{\mathcal{X}}_{et}, \mathbb{Z}/n\mathbb{Z})^D$ denotes the Pontryagin dual of the finite group $H^{2d+2-i}(\overline{\mathcal{X}}_{et}, \mathbb{Z}/n\mathbb{Z})$. The exact triangle (3) and the fact that $H^i(\overline{\mathcal{X}}_{et}, \mathbb{Q}/\mathbb{Z}(d)) = 0$ for $i < 0$ yield $H^i(\overline{\mathcal{X}}_{et}, \mathbb{Z}(d)) \simeq H^i(\overline{\mathcal{X}}_{et}, \mathbb{Q}(d))$ for $i < 0$. Hence the result for $i < 0$ follows from $H^i(\overline{\mathcal{X}}_{et}, \mathbb{Q}(d)) \simeq H^i(\mathcal{X}, \mathbb{Q}(d))$. It remains to treat the case $i = 2d+1$. Assume that $\mathcal{X}$ is flat over $\mathbb{Z}$. Then

$$H^{2d}(\mathcal{X}, \mathbb{Q}(d)) \simeq CH^d(\mathcal{X}) \otimes \mathbb{Q} = 0$$

by class field theory, hence we have

$$\begin{aligned} H^{2d+1}(\overline{\mathcal{X}}_{et}, \mathbb{Z}(d)) &\simeq H^{2d}(\overline{\mathcal{X}}_{et}, \mathbb{Q}/\mathbb{Z}(d)) \\ &\simeq \varinjlim (H^1(\overline{\mathcal{X}}_{et}, \mathbb{Z}/n\mathbb{Z})^D) \\ &\simeq (\varprojlim \mathrm{Hom}(\pi_1(\overline{\mathcal{X}}_{et}), \mathbb{Z}/n\mathbb{Z}))^D \\ &= \mathrm{Hom}(\pi_1(\overline{\mathcal{X}}_{et})^{ab}, \widehat{\mathbb{Z}})^D \\ &= 0 \end{aligned}$$

since the abelian fundamental group $\pi_1(\overline{\mathcal{X}}_{et})^{ab}$ is finite. Assume now that $\mathcal{X}$ is a smooth proper scheme over a finite field. Here the map

$$CH^{2d}(\mathcal{X}) \to H^{2d}(\mathcal{X}_{et}, \mathbb{Z}(d))$$

is an isomorphism (see for example [15] Theorem 88), and this group is finitely generated of rank one, again by class field theory. The exact triangle (3) yields an exact sequence

$$\to CH^{2d}(\mathcal{X}) \to CH^{2d}(\mathcal{X}) \otimes \mathbb{Q} \to H^{2d}(\mathcal{X}_{et}, \mathbb{Q}/\mathbb{Z}(d)) \to H^{2d+1}(\mathcal{X}_{et}, \mathbb{Z}(d)) \to 0$$

where the central map is surjective, as can be seen from

$$H^{2d}(\mathcal{X}_{et}, \mathbb{Q}/\mathbb{Z}(d)) \simeq \varinjlim (H^1(\mathcal{X}_{et}, \mathbb{Z}/n\mathbb{Z})^*) \simeq \mathbb{Q}/\mathbb{Z}$$

$\square$

Definition 4.2. *We denote by* $\mathrm{R}\Gamma(\overline{\mathcal{X}}_{et}, \mathbb{Z}(d))_{\geq 0}$ *the truncation of the complex* $\mathrm{R}\Gamma(\overline{\mathcal{X}}_{et}, \mathbb{Z}(d))$ *so that we have*

$$\begin{aligned} H^j(\overline{\mathcal{X}}_{et}, \mathbb{Z}(d))_{\geq 0} &= H^j(\overline{\mathcal{X}}_{et}, \mathbb{Z}(d)) \textit{ for } j \geq 0 \\ &= 0 \textit{ for } j < 0. \end{aligned}$$

Note that the Beilinson-Soulé vanishing conjecture with $\mathbb{Q}$-coefficients would imply that the natural map

$$\mathrm{R}\Gamma(\overline{\mathcal{X}}_{et}, \mathbb{Z}(d)) \to \mathrm{R}\Gamma(\overline{\mathcal{X}}_{et}, \mathbb{Z}(d))_{\geq 0} \tag{7}$$

is a quasi-isomorphism, since one has $H^i(\overline{\mathcal{X}}_{et}, \mathbb{Z}(d)) \simeq H^i(\mathcal{X}, \mathbb{Q}(d))$ for $i < 0$. In particular Conjecture 3.1 implies that the map (7) is a quasi-isomorphism.

Lemma 4.3. *Assume that $\mathcal{X}$ satisfies $\mathbf{L}(\overline{\mathcal{X}}_{et}, d)_{\geq 0}$. Then the natural map*

$$H^i(\overline{\mathcal{X}}_{et}, \mathbb{Z}) \to \mathrm{Hom}(H^{2d+2-i}(\overline{\mathcal{X}}_{et}, \mathbb{Z}(d))_{\geq 0}, \mathbb{Q}/\mathbb{Z})$$

is an isomorphism of abelian groups for $i \geq 1$.

Proof. The map of the lemma is given by the pairing

$$H^i(\overline{\mathcal{X}}_{et}, \mathbb{Z}) \times \mathrm{Ext}^{2d+2-i}_{\overline{\mathcal{X}}}(\mathbb{Z}, \mathbb{Z}(d)) \to H^{2d+2}(\overline{\mathcal{X}}_{et}, \mathbb{Z}(d)) \simeq \mathbb{Q}/\mathbb{Z}.$$

For $i = 1$ the result follows from Lemma 4.1, since one has

$$H^1(\overline{\mathcal{X}}_{et}, \mathbb{Z}) = \mathrm{Hom}_{cont}(\pi_1(\overline{\mathcal{X}}_{et}, p), \mathbb{Z}) = 0$$

because the fundamental group $\pi_1(\overline{\mathcal{X}}_{et}, p)$ of the Artin-Verdier étale topos is profinite.

The scheme $\mathcal{X}$ is normal and connected, hence $H^i(\overline{\mathcal{X}}_{et},\mathbb{Q})=H^i(\mathcal{X}_{et},\mathbb{Q})=\mathbb{Q}$, 0 for $i=0$ and $i\geq 1$ respectively. We obtain

$$H^i(\overline{\mathcal{X}}_{et},\mathbb{Z})=H^{i-1}(\overline{\mathcal{X}}_{et},\mathbb{Q}/\mathbb{Z})=\varinjlim H^{i-1}(\overline{\mathcal{X}}_{et},\mathbb{Z}/n\mathbb{Z})$$

for $i\geq 2$, since $\mathcal{X}_{et}$ is quasi-compact and quasi-separated. For any positive integer n the canonical map

$$\mathrm{Ext}^{2d+2-i}_{\overline{\mathcal{X}}}(\mathbb{Z}/n\mathbb{Z},\mathbb{Z}(d))\times H^i(\overline{\mathcal{X}}_{et},\mathbb{Z}/n\mathbb{Z})\rightarrow\mathbb{Q}/\mathbb{Z}$$

is a perfect pairing of finite groups (see [11] Theorem 7.8). The short exact sequence

$$0\rightarrow\mathbb{Z}\rightarrow\mathbb{Z}\rightarrow\mathbb{Z}/n\mathbb{Z}\rightarrow 0$$

yields a long exact sequence

$$H^{j-1}(\overline{\mathcal{X}}_{et},\mathbb{Z}(d))\rightarrow H^{j-1}(\overline{\mathcal{X}}_{et},\mathbb{Z}(d))\rightarrow\mathrm{Ext}^{j}_{\overline{\mathcal{X}}}(\mathbb{Z}/n\mathbb{Z},\mathbb{Z}(d))$$
$$\rightarrow H^{j}(\overline{\mathcal{X}}_{et},\mathbb{Z}(d))\rightarrow H^{j}(\overline{\mathcal{X}}_{et},\mathbb{Z}(d))$$

We obtain a short exact sequence

$$0\rightarrow H^{j-1}(\overline{\mathcal{X}}_{et},\mathbb{Z}(d))_n\rightarrow\mathrm{Ext}^{j}_{\overline{\mathcal{X}}}(\mathbb{Z}/n\mathbb{Z},\mathbb{Z}(d))\rightarrow {}_nH^{j}(\overline{\mathcal{X}}_{et},\mathbb{Z}(d))\rightarrow 0$$

for any j. By left exactness of projective limits the sequence

$$0\rightarrow\varprojlim H^{j-1}(\overline{\mathcal{X}}_{et},\mathbb{Z}(d))_n\rightarrow\varprojlim\mathrm{Ext}^{j}_{\overline{\mathcal{X}}}(\mathbb{Z}/n\mathbb{Z},\mathbb{Z}(d))\rightarrow\varprojlim {}_nH^{j}(\overline{\mathcal{X}}_{et},\mathbb{Z}(d))$$

is exact. The module $\varprojlim {}_nH^{j}(\overline{\mathcal{X}}_{et},\mathbb{Z}(d))$ vanishes for $0\leq j\leq 2d+1$ since $H^{j}(\overline{\mathcal{X}}_{et},\mathbb{Z}(d))$ is assumed to be finitely generated for such j. We have $\varprojlim {}_nH^{j}(\overline{\mathcal{X}}_{et},\mathbb{Z}(d))=0$ for $j<0$ since $H^{j}(\overline{\mathcal{X}}_{et},\mathbb{Z}(d))$ is uniquely divisible for $j<0$. This yields an isomorphism of profinite groups

$$\varprojlim H^{j-1}(\overline{\mathcal{X}}_{et},\mathbb{Z}(d))_n\xrightarrow{\sim}\varprojlim\mathrm{Ext}^{j}_{\overline{\mathcal{X}}}(\mathbb{Z}/n\mathbb{Z},\mathbb{Z}(d))\xrightarrow{\sim}(\varinjlim H^{2d+2-j}(\overline{\mathcal{X}}_{et},\mathbb{Z}/n\mathbb{Z}))^D,$$

where the last isomorphism follows from the duality above (and from the fact that $\mathrm{Ext}^{j}_{\overline{\mathcal{X}}}(\mathbb{Z}/n\mathbb{Z},\mathbb{Z}(d))$ is finite for any n). This gives isomorphisms of torsion groups

$$H^{2d+2-(j-1)}(\overline{\mathcal{X}}_{et},\mathbb{Z})\xrightarrow{\sim}(\varinjlim H^{2d+2-j}(\overline{\mathcal{X}}_{et},\mathbb{Z}/n\mathbb{Z}))^{DD}$$
$$\xrightarrow{\sim}(\varprojlim H^{j-1}(\overline{\mathcal{X}}_{et},\mathbb{Z}(d))_n)^D\xrightarrow{\sim}(H^{j-1}(\overline{\mathcal{X}}_{et},\mathbb{Z}(d))_{\geq 0})^D$$

for any $j\leq 2d+1$ (note that $2d+2-(j-1)\geq 2\Leftrightarrow j\leq 2d+1$). The last isomorphism above follows from the fact that $H^{j}(\overline{\mathcal{X}}_{et},\mathbb{Z}(d))$ is finitely generated for $0\leq j\leq 2d$ and uniquely divisible for $j<0$. Hence for any $i\geq 2$ the natural map

$$H^i(\overline{\mathcal{X}}_{et},\mathbb{Z})\rightarrow\mathrm{Hom}(H^{2d+2-i}(\overline{\mathcal{X}}_{et},\mathbb{Z}(d))_{\geq 0},\mathbb{Q}/\mathbb{Z})$$

is an isomorphism. □

Theorem 4.4. *Assume that $\mathcal{X}$ satisfies $\mathbf{L}(\overline{\mathcal{X}}_{et},d)_{\geq 0}$. There is a canonical morphism in $\mathcal{D}$:*

$$\alpha_{\mathcal{X}}:\mathrm{RHom}(\mathrm{R}\Gamma(\mathcal{X},\mathbb{Q}(d))_{\geq 0},\mathbb{Q}[-2d-2])\rightarrow\mathrm{R}\Gamma(\overline{\mathcal{X}}_{et},\mathbb{Z})$$

functorial in $\mathcal{X}$ and such that $H^i(\alpha_{\mathcal{X}})$ factors as follows

$$\mathrm{Hom}(H^{2d+2-i}(\mathcal{X},\mathbb{Q}(d))_{\geq 0},\mathbb{Q})\twoheadrightarrow H^i(\overline{\mathcal{X}}_{et},\mathbb{Z})_{div}\hookrightarrow H^i(\overline{\mathcal{X}}_{et},\mathbb{Z})$$

where $H^i(\overline{\mathcal{X}}_{et}, \mathbb{Z})_{div}$ denotes the maximal divisible subgroup of $H^i(\overline{\mathcal{X}}_{et}, \mathbb{Z})$.

Proof. We set $\delta := 2d + 2$. There is a natural map in $\mathcal{D}$:

$$\mathrm{RHom}(\mathbb{Z}, \mathbb{Z}(d)) \to \mathrm{RHom}(\mathrm{R}\Gamma(\overline{\mathcal{X}}_{et}, \mathbb{Z}), \mathrm{R}\Gamma(\overline{\mathcal{X}}_{et}, \mathbb{Z}(d))) \to \mathrm{RHom}(\mathrm{R}\Gamma(\overline{\mathcal{X}}_{et}, \mathbb{Z}), \mathbb{Q}/\mathbb{Z}[-\delta])$$

where the second map is induced by

$$\mathrm{R}\Gamma(\overline{\mathcal{X}}_{et}, \mathbb{Z}(d)) \to \mathrm{R}\Gamma(\overline{\mathcal{X}}_{et}, \mathbb{Z}(d))_{\geq \delta} \simeq \mathbb{Q}/\mathbb{Z}[-\delta].$$

Hence we have

$$\mathrm{R}\Gamma(\overline{\mathcal{X}}_{et}, \mathbb{Z}(d)) \to \mathrm{RHom}(\mathrm{R}\Gamma(\overline{\mathcal{X}}_{et}, \mathbb{Z}), \mathbb{Q}/\mathbb{Z}[-\delta]) \tag{8}$$

The complex $\mathrm{RHom}(\mathrm{R}\Gamma(\overline{\mathcal{X}}_{et}, \mathbb{Z}), \mathbb{Q}/\mathbb{Z}[-\delta])$ is acyclic in negative degrees hence (8) induces

$$\mathrm{R}\Gamma(\overline{\mathcal{X}}_{et}, \mathbb{Z}(d))_{\geq 0} \to \mathrm{RHom}(\mathrm{R}\Gamma(\overline{\mathcal{X}}_{et}, \mathbb{Z}), \mathbb{Q}/\mathbb{Z}[-\delta])$$

Applying the functor $\mathrm{RHom}(-, \mathbb{Q}/\mathbb{Z}[-\delta])$ we obtain

$$\mathrm{RHom}(\mathrm{RHom}(\mathrm{R}\Gamma(\overline{\mathcal{X}}_{et}, \mathbb{Z}), \mathbb{Q}/\mathbb{Z}[-\delta]), \mathbb{Q}/\mathbb{Z}[-\delta]) \to \mathrm{RHom}(\mathrm{R}\Gamma(\overline{\mathcal{X}}_{et}, \mathbb{Z}(d))_{\geq 0}, \mathbb{Q}/\mathbb{Z}[-\delta]).$$

Composing the previous morphism with

$$\mathrm{R}\Gamma(\overline{\mathcal{X}}_{et}, \mathbb{Z}) \to \mathrm{RHom}(\mathrm{RHom}(\mathrm{R}\Gamma(\overline{\mathcal{X}}_{et}, \mathbb{Z}), \mathbb{Q}/\mathbb{Z}[-\delta]), \mathbb{Q}/\mathbb{Z}[-\delta])$$

we get

$$\mathrm{R}\Gamma(\overline{\mathcal{X}}_{et}, \mathbb{Z}) \to \mathrm{RHom}(\mathrm{R}\Gamma(\overline{\mathcal{X}}_{et}, \mathbb{Z}(d))_{\geq 0}, \mathbb{Q}/\mathbb{Z}[-\delta]). \tag{9}$$

Consider the following diagram

$$\begin{tikzcd}
 & \mathrm{RHom}(\mathrm{R}\Gamma(\mathcal{X}, \mathbb{Q}(d))_{\geq 0}, \mathbb{Q}[-\delta]) \\
 & \mathrm{RHom}(\mathrm{R}\Gamma(\overline{\mathcal{X}}_{et}, \mathbb{Z}(d))_{\geq 0} \otimes \mathbb{Q}, \mathbb{Q}[-\delta]) \arrow[u, "a", "\simeq"'] \arrow[d, "b"', "\simeq"] \\
 & \mathrm{RHom}(\mathrm{R}\Gamma(\overline{\mathcal{X}}_{et}, \mathbb{Z}(d))_{\geq 0}, \mathbb{Q}[-\delta]) \arrow[d, "c"'] \\
\mathrm{R}\Gamma(\overline{\mathcal{X}}_{et}, \mathbb{Z}) \arrow[r] & \mathrm{RHom}(\mathrm{R}\Gamma(\overline{\mathcal{X}}_{et}, \mathbb{Z}(d))_{\geq 0}, \mathbb{Q}/\mathbb{Z}[-\delta])
\end{tikzcd}$$

The map a is a quasi-isomorphism since étale motivic cohomology and motivic cohomology agree with rational coefficients. The map b is clearly a quasi-isomorphism. The map c is induced by the quotient morphism $\mathbb{Q}[-\delta] \to \mathbb{Q}/\mathbb{Z}[-\delta]$. By Proposition 4.3 the map (9) sits in an exact triangle

$$\mathrm{R}\Gamma(\overline{\mathcal{X}}_{et}, \mathbb{Z}) \to \mathrm{RHom}(\mathrm{R}\Gamma(\overline{\mathcal{X}}_{et}, \mathbb{Z}(d))_{\geq 0}, \mathbb{Q}/\mathbb{Z}[-\delta]) \to \widehat{\mathbb{Z}}/\mathbb{Z}[0] \to \mathrm{R}\Gamma(\overline{\mathcal{X}}_{et}, \mathbb{Z})[1]$$

We set $D_{\mathcal{X}} := \mathrm{RHom}(\mathrm{R}\Gamma(\mathcal{X}, \mathbb{Q}(d))_{\geq 0}, \mathbb{Q}[-\delta])$. We obtain an exact sequence of abelian groups

$$\mathrm{Hom}_{\mathcal{D}}(D_{\mathcal{X}}, \widehat{\mathbb{Z}}/\mathbb{Z}[-1]) \to \mathrm{Hom}_{\mathcal{D}}(D_{\mathcal{X}}, \mathrm{R}\Gamma(\overline{\mathcal{X}}_{et}, \mathbb{Z}))$$
$$\to \mathrm{Hom}_{\mathcal{D}}(D_{\mathcal{X}}, \mathrm{RHom}(\mathrm{R}\Gamma(\overline{\mathcal{X}}_{et}, \mathbb{Z}(d))_{\geq 0}, \mathbb{Q}/\mathbb{Z}[-\delta])) \to \mathrm{Hom}_{\mathcal{D}}(D_{\mathcal{X}}, \widehat{\mathbb{Z}}/\mathbb{Z}[0])$$

where $\widehat{\mathbb{Z}} := \varprojlim \mathbb{Z}/n\mathbb{Z}$. But $\widehat{\mathbb{Z}}/\mathbb{Z}[i]$ is a bounded complex of injective abelian groups for $i = -1, 0$ so that

$$\mathrm{Hom}_{\mathcal{D}}(D_{\mathcal{X}}, \widehat{\mathbb{Z}}/\mathbb{Z}[i]) = \mathrm{Hom}_{\mathcal{K}}(D_{\mathcal{X}}, \widehat{\mathbb{Z}}/\mathbb{Z}[i])$$

where $\mathcal{K}$ is the category of cochain complexes of abelian groups modulo homotopy. Indeed, the surjective map $\mathbb{V} \twoheadrightarrow \widehat{\mathbb{Z}}/\mathbb{Z}$, where $\mathbb{V} = \mathrm{Hom}(\mathbb{Q}, \mathbb{S}^1)$ is the solenoid, shows that $\widehat{\mathbb{Z}}/\mathbb{Z}$ is divisible.

The facts that $D_{\mathcal{X}}$ is acyclic in degrees < 2 (since $H^j(\mathcal{X}, \mathbb{Q}(d)) = 0$ for $j > 2d$) and that $\widehat{\mathbb{Z}}/\mathbb{Z}[i]$ is concentrated in degree $-i = 0, 1$ imply

$$\mathrm{Hom}_{\mathcal{D}}(D_{\mathcal{X}}, \widehat{\mathbb{Z}}/\mathbb{Z}[i]) = \mathrm{Hom}_{\mathcal{K}}(D_{\mathcal{X}}, \widehat{\mathbb{Z}}/\mathbb{Z}[i]) = 0.$$

The natural map (given by composition with (9))

$$\mathrm{Hom}_{\mathcal{D}}(D_{\mathcal{X}}, \mathrm{R}\Gamma(\overline{\mathcal{X}}_{et}, \mathbb{Z})) \to \mathrm{Hom}_{\mathcal{D}}(D_{\mathcal{X}}, \mathrm{RHom}(\mathrm{R}\Gamma(\overline{\mathcal{X}}_{et}, \mathbb{Z}(d))_{\geq 0}, \mathbb{Q}/\mathbb{Z}[-\delta]))$$

is therefore an isomorphism, hence there exists a *unique* morphism in $\mathcal{D}$

$$\alpha_{\mathcal{X}} : D_{\mathcal{X}} = \mathrm{RHom}(\mathrm{R}\Gamma(\mathcal{X}, \mathbb{Q}(d))_{\geq 0}, \mathbb{Q}[-2d-2]) \to \mathrm{R}\Gamma(\overline{\mathcal{X}}_{et}, \mathbb{Z})$$

which makes the following diagram commutative:

$$\begin{tikzcd}
& \mathrm{RHom}(\mathrm{R}\Gamma(\mathcal{X}, \mathbb{Q}(d))_{\geq 0}, \mathbb{Q}[-\delta]) \arrow[dl, "\exists !\ \alpha_{\mathcal{X}}"'] \arrow[d] \\
\mathrm{R}\Gamma(\overline{\mathcal{X}}_{et}, \mathbb{Z}) \arrow[r] & \mathrm{RHom}(\mathrm{R}\Gamma(\overline{\mathcal{X}}_{et}, \mathbb{Z}(d))_{\geq 0}, \mathbb{Q}/\mathbb{Z}[-\delta])
\end{tikzcd}$$

Here the vertical map is the morphism defined by the previous commutative diagram. The horizontal map is the morphism (9). The functorial behavior of $\alpha_{\mathcal{X}}$ is given by Theorem 5.4. □

4.2. **The case $\mathcal{X}(\mathbb{R}) \neq \emptyset$.** An Artin-Verdier duality theorem in the spirit of [1] would make the previous proof valid in the case $\mathcal{X}(\mathbb{R}) \neq \emptyset$. Such a result is not currently available in the literature so we use cohomology with compact support in the sense of [23].

Proposition 4.5. *Theorem 4.4 holds for any proper regular connected arithmetic scheme $\mathcal{X}$ satisfying $\mathbf{L}(\overline{\mathcal{X}}_{et}, d)_{\geq 0}$.*

Proof. Let $S = \mathrm{Spec}(\mathcal{O}_{\mathcal{X}}(\mathcal{X}))$ and let $f : \mathcal{X} \to S$ be the canonical map. Then $\mathcal{X}$ is proper over the number ring S. We denote by

$$R\hat{\Gamma}_c(\mathcal{X}_{et}, \mathcal{F}) := R\hat{\Gamma}_c(S_{et}, Rf_*\mathcal{F})$$

the "cohomology with compact support", where $R\hat{\Gamma}_c(S_{et}, -)$ is defined as in [23]. We consider the map

$$\mathrm{RHom}_{\mathcal{X}}(\mathbb{Z}, \mathbb{Z}(d)) \to \mathrm{RHom}_S(Rf_*\mathbb{Z}, Rf_*\mathbb{Z}(d)) \xrightarrow{p} \mathrm{RHom}_S(Rf_*\mathbb{Z}, \mathbb{Z}(1)[-2d+2])$$

$$\xrightarrow{d} \mathrm{RHom}_{\mathcal{D}}(R\hat{\Gamma}_c(S_{et}, Rf_*\mathbb{Z}), \mathbb{Q}/\mathbb{Z}[-2d-2]) = \mathrm{RHom}_{\mathcal{D}}(R\hat{\Gamma}_c(\mathcal{X}_{et}, \mathbb{Z}), \mathbb{Q}/\mathbb{Z}[-\delta])$$

where p is induced by the push-forward map of [11] Corollary 7.2 (b) and d is given by 1-dimensional Artin-Verdier duality [23]. This map induces

$$\mathrm{R}\Gamma(\mathcal{X}_{et}, \mathbb{Z}(d))_{\geq 0} \to \mathrm{RHom}(\mathrm{R}\hat{\Gamma}_c(\mathcal{X}_{et}, \mathbb{Z}), \mathbb{Q}/\mathbb{Z}[-\delta]).$$

since $\mathrm{R}\hat{\Gamma}_c(\mathcal{X}_{et},\mathbb{Z})$ is acyclic in degrees $>\delta$. We obtain the morphism

$$\mathrm{R}\hat{\Gamma}_c(\mathcal{X}_{et},\mathbb{Z})\rightarrow \mathrm{RHom}(\mathrm{R}\Gamma(\mathcal{X}_{et},\mathbb{Z}(d))_{\geq 0},\mathbb{Q}/\mathbb{Z}[-\delta]). \tag{10}$$

Then we show that

$$\hat{H}^i_c(\mathcal{X}_{et},\mathbb{Z})\rightarrow \mathrm{Hom}(H^{2d+2-i}(\mathcal{X}_{et},\mathbb{Z}(d))_{\geq 0},\mathbb{Q}/\mathbb{Z})$$

is an isomorphism of abelian groups for $i\neq 0$ and that the map

$$\hat{H}^0_c(\mathcal{X}_{et},\mathbb{Z})\rightarrow \mathrm{Hom}(H^{2d+2}(\mathcal{X}_{et},\mathbb{Z}(d)),\mathbb{Q}/\mathbb{Z})$$

is injective with divisible cokernel. The same argument as in the proof of Theorem 4.4 provides us with a commutative diagram

$$\begin{tikzcd}
 & \mathrm{RHom}(\mathrm{R}\Gamma(\mathcal{X},\mathbb{Q}(d))_{\geq 0},\mathbb{Q}[-\delta]) \arrow[dl, "\exists!\,\tilde{\alpha}_{\mathcal{X}}"'] \arrow[d] \\
\mathrm{R}\hat{\Gamma}_c(\mathcal{X}_{et},\mathbb{Z}) \arrow[r] & \mathrm{RHom}(\mathrm{R}\Gamma(\mathcal{X}_{et},\mathbb{Z}(d))_{\geq 0},\mathbb{Q}/\mathbb{Z}[-\delta])
\end{tikzcd}$$

Composing $\tilde{\alpha}_{\mathcal{X}}$ with

$$\mathrm{R}\hat{\Gamma}_c(\mathcal{X}_{et},\mathbb{Z}):=\mathrm{R}\hat{\Gamma}_c(S_{et},Rf_*\mathbb{Z})\rightarrow \mathrm{R}\Gamma(S_{et},Rf_*\mathbb{Z})\simeq \mathrm{R}\Gamma(\mathcal{X}_{et},\mathbb{Z})$$

we obtain the map

$$D_{\mathcal{X}}=\mathrm{RHom}(\mathrm{R}\Gamma(\mathcal{X},\mathbb{Q}(d))_{\geq 0},\mathbb{Q}[-\delta])\rightarrow \mathrm{R}\Gamma(\mathcal{X}_{et},\mathbb{Z})$$

Consider now the exact triangle

$$\mathrm{R}\Gamma_{\mathcal{X}_\infty}(\overline{\mathcal{X}}_{et},\mathbb{Z})\rightarrow \mathrm{R}\Gamma(\overline{\mathcal{X}}_{et},\mathbb{Z})\rightarrow \mathrm{R}\Gamma(\mathcal{X}_{et},\mathbb{Z})$$

Then for any n the group

$$H^n_{\mathcal{X}_\infty}(\overline{\mathcal{X}}_{et},\mathbb{Z}):=H^n(\mathrm{R}\Gamma_{\mathcal{X}_\infty}(\overline{\mathcal{X}}_{et},\mathbb{Z}))$$

is killed by 2. It follows that

$$\mathrm{Hom}_{\mathcal{D}}(D_{\mathcal{X}},\mathrm{R}\Gamma_{\mathcal{X}_\infty}(\overline{\mathcal{X}}_{et},\mathbb{Z}))=\mathrm{Hom}_{\mathcal{D}}(D_{\mathcal{X}},\mathrm{R}\Gamma_{\mathcal{X}_\infty}(\overline{\mathcal{X}}_{et},\mathbb{Z})[1])=0$$

since $D_{\mathcal{X}}$ is divisible. Hence there exists a unique map

$$\alpha_{\mathcal{X}}:\mathrm{RHom}(\mathrm{R}\Gamma(\mathcal{X},\mathbb{Q}(d))_{\geq 0},\mathbb{Q}[-\delta])\rightarrow \mathrm{R}\Gamma(\overline{\mathcal{X}}_{et},\mathbb{Z})$$

such that the diagram

$$\begin{tikzcd}
 & \mathrm{RHom}(\mathrm{R}\Gamma(\mathcal{X},\mathbb{Q}(d))_{\geq 0},\mathbb{Q}[-\delta]) \arrow[dl, "\exists!\,\alpha_{\mathcal{X}}"'] \arrow[d] \\
\mathrm{R}\Gamma(\overline{\mathcal{X}}_{et},\mathbb{Z}) \arrow[r] & \mathrm{R}\Gamma(\mathcal{X}_{et},\mathbb{Z})
\end{tikzcd}$$

commutes. $\square$

5. Cohomology with $\mathbb{Z}$-coefficients

Throughout this section $\mathcal{X}$ denotes a proper regular connected arithmetic scheme of dimension d satisfying $\mathbf{L}(\overline{\mathcal{X}}_{et}, d)_{\geq 0}$.

Definition 5.1. *There exists an exact triangle*

$$\mathrm{RHom}(\mathrm{R}\Gamma(\mathcal{X}, \mathbb{Q}(d))_{\geq 0}, \mathbb{Q}[-\delta]) \stackrel{\alpha_{\mathcal{X}}}{\to} \mathrm{R}\Gamma(\overline{\mathcal{X}}_{et}, \mathbb{Z}) \to \mathrm{R}\Gamma_W(\overline{\mathcal{X}}, \mathbb{Z})$$
$$\to \mathrm{RHom}(\mathrm{R}\Gamma(\mathcal{X}, \mathbb{Q}(d))_{\geq 0}, \mathbb{Q}[-\delta])[1]$$

well defined up to a unique isomorphism in $\mathcal{D}$. We define

$$H^i_W(\overline{\mathcal{X}}, \mathbb{Z}) := H^i(\mathrm{R}\Gamma_W(\overline{\mathcal{X}}, \mathbb{Z})).$$

The existence of such an exact triangle follows from axiom TR1 of triangulated categories. Its uniqueness is given by Corollary 5.5 and can be stated as follows. If we have two objects $\mathrm{R}\Gamma_W(\overline{\mathcal{X}}, \mathbb{Z})$ and $\mathrm{R}\Gamma_W(\overline{\mathcal{X}}, \mathbb{Z})'$ given with exact triangles as in Definition 5.1, then there exists a *unique* map $f : \mathrm{R}\Gamma_W(\overline{\mathcal{X}}, \mathbb{Z}) \to \mathrm{R}\Gamma_W(\overline{\mathcal{X}}, \mathbb{Z})'$ in $\mathcal{D}$ which yields a morphism of exact triangles

$$\begin{tikzcd}
\mathrm{RHom}(\mathrm{R}\Gamma(\mathcal{X}, \mathbb{Q}(d))_{\geq 0}, \mathbb{Q}[-\delta]) \arrow[r, "\alpha_{\mathcal{X}}"] \arrow[d, "Id"] & \mathrm{R}\Gamma(\overline{\mathcal{X}}_{et}, \mathbb{Z}) \arrow[r] \arrow[d, "Id"] & R\Gamma(\overline{\mathcal{X}}_W, \mathbb{Z}) \arrow[r] \arrow[d, "f"] & {} \\
\mathrm{RHom}(\mathrm{R}\Gamma(\mathcal{X}, \mathbb{Q}(d))_{\geq 0}, \mathbb{Q}[-\delta]) \arrow[r, "\alpha_{\mathcal{X}}"] & R\Gamma(\overline{\mathcal{X}}_{et}, \mathbb{Z}) \arrow[r] & R\Gamma(\overline{\mathcal{X}}_W, \mathbb{Z}) \arrow[r] & {}
\end{tikzcd}$$

Lemma 5.2. *Assume that $\mathcal{X}(\mathbb{R}) = \emptyset$. There is an exact sequence*

$$0 \to H^i(\overline{\mathcal{X}}_{et}, \mathbb{Z})_{codiv} \to H^i_W(\overline{\mathcal{X}}, \mathbb{Z}) \to \mathrm{Hom}_{\mathbb{Z}}(H^{2d+2-(i+1)}(\overline{\mathcal{X}}_{et}, \mathbb{Z}(d))_{\geq 0}, \mathbb{Z}) \to 0$$

for any $i \in \mathbb{Z}$.

Proof. From the exact triangle of Defintion 5.1, we obtain a long exact sequence

$$... \to \mathrm{Hom}(H^{\delta-i}(\mathcal{X}, \mathbb{Q}(d))_{\geq 0}, \mathbb{Q}) \to H^i(\overline{\mathcal{X}}_{et}, \mathbb{Z}) \to H^i_W(\overline{\mathcal{X}}, \mathbb{Z})$$
$$\to \mathrm{Hom}(H^{\delta-(i+1)}(\mathcal{X}, \mathbb{Q}(d))_{\geq 0}, \mathbb{Q}) \to H^{i+1}(\overline{\mathcal{X}}_{et}, \mathbb{Z}) \to ...$$

which yields a short exact sequence

$$0 \to \mathrm{Coker} H^i(\alpha_{\mathcal{X}}) \to H^i_W(\overline{\mathcal{X}}, \mathbb{Z}) \to \mathrm{Ker} H^{i+1}(\alpha_{\mathcal{X}}) \to 0 \tag{11}$$

Assume for the moment that $\mathcal{X}(\mathbb{R}) = \emptyset$. For $i \geq 1$, the morphism

$$H^i(\alpha_{\mathcal{X}}) : \mathrm{Hom}(H^{\delta-i}(\mathcal{X}, \mathbb{Q}(d))_{\geq 0}, \mathbb{Q}) \to H^i(\overline{\mathcal{X}}_{et}, \mathbb{Z})$$

is the following composite map:

$$\mathrm{Hom}(H^{\delta-i}(\mathcal{X}, \mathbb{Q}(d))_{\geq 0}, \mathbb{Q}) \simeq \mathrm{Hom}(H^{\delta-i}(\overline{\mathcal{X}}_{et}, \mathbb{Z}(d))_{\geq 0}, \mathbb{Q}) \to$$
$$\to \mathrm{Hom}(H^{\delta-i}(\overline{\mathcal{X}}_{et}, \mathbb{Z}(d))_{\geq 0}, \mathbb{Q}/\mathbb{Z}) \simeq H^i(\overline{\mathcal{X}}_{et}, \mathbb{Z})$$

Since $H^j(\overline{\mathcal{X}}_{et}, \mathbb{Z}(d))_{\geq 0}$ is assumed to be finitely generated, the image of the morphism $H^i(\alpha_{\mathcal{X}})$ is the maximal divisible subgroup of $H^i(\overline{\mathcal{X}}_{et}, \mathbb{Z})$, which we denote by $H^i(\overline{\mathcal{X}}_{et}, \mathbb{Z})_{div}$. The same is true for $i \leq 0$ since one has

$$H^0(\alpha_{\mathcal{X}}) : 0 = \mathrm{Hom}(H^{\delta}(\mathcal{X}, \mathbb{Q}(d)), \mathbb{Q}) \to H^i(\overline{\mathcal{X}}_{et}, \mathbb{Z}) = \mathbb{Z}$$

and

$$H^{\delta-i}(\mathcal{X}, \mathbb{Q}(d)) = H^i(\overline{\mathcal{X}}_{et}, \mathbb{Z}) = 0 \text{ for } i < 0.$$

We obtain

$$\mathrm{Coker}H^i(\alpha_{\mathcal{X}}) = H^i(\overline{\mathcal{X}}_{et},\mathbb{Z})/H^i(\overline{\mathcal{X}}_{et},\mathbb{Z})_{div} =: H^i(\overline{\mathcal{X}}_{et},\mathbb{Z})_{codiv}$$

for any $i\in\mathbb{Z}$. The same is true for $\mathcal{X}(\mathbb{R})\neq\emptyset$. The kernel of $H^i(\alpha_{\mathcal{X}})$ is given by the following commutative diagram with exact rows:

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{Ker}H^i(\alpha_{\mathcal{X}}) \arrow[r] & \mathrm{Hom}(H^{\delta-i}(\mathcal{X},\mathbb{Q}(d))_{\geq 0},\mathbb{Q}) \arrow[r] & \mathrm{Hom}(H^{\delta-i}(\overline{\mathcal{X}}_{et},\mathbb{Z}(d))_{\geq 0},\mathbb{Q}/\mathbb{Z}) \\
0 \arrow[r] & \mathrm{Hom}(H^{\delta-i}(\overline{\mathcal{X}}_{et},\mathbb{Z}(d))_{\geq 0},\mathbb{Z}) \arrow[r] & \mathrm{Hom}(H^{\delta-i}(\overline{\mathcal{X}}_{et},\mathbb{Z}(d))_{\geq 0},\mathbb{Q}) \arrow[r] \arrow[u, "\simeq"] & \mathrm{Hom}(H^{\delta-i}(\overline{\mathcal{X}}_{et},\mathbb{Z}(d))_{\geq 0},\mathbb{Q}/\mathbb{Z}) \arrow[u, "="']
\end{tikzcd}$$

The exact sequence (11) therefore takes the form

$$0\to H^i(\overline{\mathcal{X}}_{et},\mathbb{Z})_{codiv}\to H^i_W(\overline{\mathcal{X}},\mathbb{Z})\to \mathrm{Hom}_{\mathbb{Z}}(H^{2d+2-(i+1)}(\overline{\mathcal{X}}_{et},\mathbb{Z}(d)),\mathbb{Z})\to 0$$

□

Proposition 5.3. *The group $H^i_W(\overline{\mathcal{X}},\mathbb{Z})$ is finitely generated for any $i\in\mathbb{Z}$, and one has $H^i_W(\overline{\mathcal{X}},\mathbb{Z})=0$ for $i<0$, $H^0_W(\overline{\mathcal{X}},\mathbb{Z})=\mathbb{Z}$ and $H^i_W(\overline{\mathcal{X}},\mathbb{Z})=0$ for i large.*

Proof. Assume that $\mathcal{X}(\mathbb{R})=\emptyset$ and consider the exact sequence

$$0\to H^i(\overline{\mathcal{X}}_{et},\mathbb{Z})_{codiv}\to H^i_W(\overline{\mathcal{X}},\mathbb{Z})\to \mathrm{Hom}_{\mathbb{Z}}(H^{2d+2-(i+1)}(\overline{\mathcal{X}}_{et},\mathbb{Z}(d))_{\geq 0},\mathbb{Z})\to 0$$

One has $H^i(\overline{\mathcal{X}}_{et},\mathbb{Z})=0$ for $i<0$, $H^0(\overline{\mathcal{X}}_{et},\mathbb{Z})=\mathbb{Z}$ and $H^j(\overline{\mathcal{X}}_{et},\mathbb{Z}(d))$ is torsion for $j>2d$. The result for $i\leq 0$ follows. Let $i\geq 1$. The isomorphism

$$H^i(\overline{\mathcal{X}}_{et},\mathbb{Z})\simeq \mathrm{Hom}(H^{2d+2-i}(\overline{\mathcal{X}}_{et},\mathbb{Z}(d))_{\geq 0},\mathbb{Q}/\mathbb{Z})$$

and the fact that $H^{2d+2-i}(\overline{\mathcal{X}}_{et},\mathbb{Z}(d))_{\geq 0}$ is assumed to be finitely generated imply that $H^i(\overline{\mathcal{X}}_{et},\mathbb{Z})_{codiv}$ is finite and that $\mathrm{Hom}_{\mathbb{Z}}(H^{2d+2-(i+1)}(\overline{\mathcal{X}}_{et},\mathbb{Z}(d))_{\geq 0},\mathbb{Z})$ is finitely generated.

For $\mathcal{X}(\mathbb{R})\neq\emptyset$ we have an exact sequence

$$0\to H^i(\overline{\mathcal{X}}_{et},\mathbb{Z})_{codiv}\to H^i_W(\overline{\mathcal{X}},\mathbb{Z})\to \mathrm{Ker}H^{i+1}(\alpha_{\mathcal{X}})\to 0$$

where $\mathrm{Ker}H^{i+1}(\alpha_{\mathcal{X}})$ is a lattice in $\mathrm{Hom}(H^{2d+2-(i+1)}(\mathcal{X},\mathbb{Q}(d))_{\geq 0},\mathbb{Q})$. It follows that $H^i_W(\overline{\mathcal{X}},\mathbb{Z})$ is finitely generated for any i and vanishes for $i<0$. Moreover one has $H^0_W(\overline{\mathcal{X}},\mathbb{Z})=\mathbb{Z}$. For i large we have $H^{2d+2-(i+1)}(\mathcal{X},\mathbb{Q}(d))_{\geq 0}=0$ and $H^i(\overline{\mathcal{X}}_{et},\mathbb{Z})=0$. Hence $H^i_W(\overline{\mathcal{X}},\mathbb{Z})=0$ for i large. □

Theorem 5.4. *Let $f:\mathcal{X}\to\mathcal{Y}$ be a morphism of relative dimension c between proper regular connected arithmetic schemes, such that $\mathbf{L}(\overline{\mathcal{X}}_{et},d_{\mathcal{X}})_{\geq 0}$ and $\mathbf{L}(\overline{\mathcal{Y}}_{et},d_{\mathcal{Y}})_{\geq 0}$ hold. We choose complexes $R\Gamma_W(\overline{\mathcal{X}},\mathbb{Z})$ and $R\Gamma_W(\overline{\mathcal{Y}},\mathbb{Z})$ as in Definition 5.1. Assume that either $c\geq 0$ or $\mathbf{L}(\overline{\mathcal{X}}_{et},d_{\mathcal{X}})$ holds.*

Then there exists a unique *map in $\mathcal{D}$*

$$R\Gamma_W(\overline{\mathcal{Y}},\mathbb{Z})\to R\Gamma_W(\overline{\mathcal{X}},\mathbb{Z})$$

which sits in a morphism of exact triangles

$$\begin{tikzcd}
\mathrm{RHom}(R\Gamma(\mathcal{X},\mathbb{Q}(d_{\mathcal{X}}))_{\geq 0},\mathbb{Q}[-\delta_{\mathcal{X}}]) \arrow[r] & R\Gamma(\overline{\mathcal{X}}_{et},\mathbb{Z}) \arrow[r] & R\Gamma_W(\overline{\mathcal{X}},\mathbb{Z}) \arrow[r] & {} \\
\mathrm{RHom}(R\Gamma(\mathcal{Y},\mathbb{Q}(d_{\mathcal{Y}}))_{\geq 0},\mathbb{Q}[-\delta_{\mathcal{Y}}]) \arrow[r] \arrow[u] & R\Gamma(\overline{\mathcal{Y}}_{et},\mathbb{Z}) \arrow[r] \arrow[u, "f^*_{et}"'] & R\Gamma_W(\overline{\mathcal{Y}},\mathbb{Z}) \arrow[r] \arrow[u, "f^*_W"'] & {}
\end{tikzcd}$$

Proof. Let $\mathcal{X}$ and $\mathcal{Y}$ be connected, proper and regular arithmetic schemes of dimension $d_{\mathcal{X}}$ and $d_{\mathcal{Y}}$ respectively. We choose complexes $\mathrm{R}\Gamma_W(\overline{\mathcal{X}},\mathbb{Z})$ and $\mathrm{R}\Gamma_W(\overline{\mathcal{Y}},\mathbb{Z})$ as in Def. 5.1. Let $f:\mathcal{X}\to\mathcal{Y}$ be a morphism of relative dimension $c=d_{\mathcal{X}}-d_{\mathcal{Y}}$. The morphism f is proper and the map

$$z^n(\mathcal{X},*)\to z^{n-c}(\mathcal{Y},*)$$

induces a morphism

$$f_*\mathbb{Q}(d_{\mathcal{X}})\to\mathbb{Q}(d_{\mathcal{Y}}))[-2c]$$

of complexes of abelian Zariski sheaves on $\mathcal{Y}$. We need to see that

$$f_*\mathbb{Q}(d_{\mathcal{X}})\simeq \mathrm{R}f_*\mathbb{Q}(d_{\mathcal{X}}) \tag{12}$$

Localizing over the base, it is enough to show this fact for f a proper map over a discrete valuation ring. By [20], higher Chow groups satisfy localization for schemes of finite type over a discrete valuation ring, hence the Mayer-Vietoris property. It follows that, over a discrete valuation ring, the hypercohomology of the cycle complex coincides with its cohomology (as a complex of abelian group). This yields (12) and a morphism of complexes

$$R\Gamma(\mathcal{X},\mathbb{Q}(d_{\mathcal{X}}))\simeq R\Gamma(\mathcal{Y},f_*\mathbb{Q}(d_{\mathcal{X}}))\to R\Gamma(\mathcal{Y},\mathbb{Q}(d_{\mathcal{Y}}))[-2c]$$

If $c\geq 0$ this induces a morphism

$$R\Gamma(\mathcal{X},\mathbb{Q}(d_{\mathcal{X}}))_{\geq 0}\to R\Gamma(\mathcal{Y},\mathbb{Q}(d_{\mathcal{Y}}))_{\geq 0}[-2c]$$

If $c<0$ then $\mathbf{L}(\overline{\mathcal{X}}_{et},d_{\mathcal{X}})$ holds by assumption, and we consider the map

$$R\Gamma(\mathcal{X},\mathbb{Q}(d_{\mathcal{X}}))_{\geq 0}\xleftarrow{\sim} R\Gamma(\mathcal{X},\mathbb{Q}(d_{\mathcal{X}}))\to R\Gamma(\mathcal{Y},\mathbb{Q}(d_{\mathcal{Y}}))[-2c]\to R\Gamma(\mathcal{Y},\mathbb{Q}(d_{\mathcal{Y}}))_{\geq 0}[-2c]$$

where the first isomorphism is given by Proposition 3.4. In both cases, we get a morphism

$$\mathrm{RHom}(R\Gamma(\mathcal{Y},\mathbb{Q}(d_{\mathcal{Y}}))_{\geq 0},\mathbb{Q}[-\delta_{\mathcal{Y}}])\to \mathrm{RHom}(R\Gamma(\mathcal{X},\mathbb{Q}(d_{\mathcal{X}}))_{\geq 0},\mathbb{Q}[-\delta_{\mathcal{X}}])$$

such that the following square is commutative:

$$\begin{array}{ccc}
\mathrm{RHom}(R\Gamma(\mathcal{Y},\mathbb{Q}(d_{\mathcal{Y}}))_{\geq 0},\mathbb{Q}[-\delta_{\mathcal{Y}}]) & \xrightarrow{\alpha_{\mathcal{Y}}} & \mathrm{R}\Gamma(\overline{\mathcal{Y}}_{et},\mathbb{Z}) \\
\downarrow & & \downarrow \\
\mathrm{RHom}(R\Gamma(\mathcal{X},\mathbb{Q}(d_{\mathcal{X}}))_{\geq 0},\mathbb{Q}[-\delta_{\mathcal{X}}]) & \xrightarrow{\alpha_{\mathcal{X}}} & \mathrm{R}\Gamma(\overline{\mathcal{X}}_{et},\mathbb{Z})
\end{array}$$

Hence there exists a morphism

$$f^*_W:\mathrm{R}\Gamma_W(\overline{\mathcal{Y}},\mathbb{Z})\to\mathrm{R}\Gamma_W(\overline{\mathcal{X}},\mathbb{Z})$$

inducing an morphism of exact triangles. We claim that such a morphism f^*_W is unique. In order to ease the notations, we set

$$D_{\mathcal{X}}:=\mathrm{RHom}(R\Gamma(\mathcal{X},\mathbb{Q}(d_{\mathcal{X}}))_{\geq 0},\mathbb{Q}[-\delta_{\mathcal{X}}])\text{ and }D_{\mathcal{Y}}:=\mathrm{RHom}(R\Gamma(\mathcal{Y},\mathbb{Q}(d_{\mathcal{Y}}))_{\geq 0},\mathbb{Q}[-\delta_{\mathcal{Y}}]).$$

The complexes $\mathrm{R}\Gamma_W(\overline{\mathcal{X}},\mathbb{Z})$ and $\mathrm{R}\Gamma_W(\overline{\mathcal{Y}},\mathbb{Z})$ are both perfect complexes of abelian groups, since they are bounded complexes with finitely generated cohomology groups. Applying the functor $\mathrm{Hom}_{\mathcal{D}}(-,\mathrm{R}\Gamma_W(\overline{\mathcal{X}},\mathbb{Z}))$ to the exact triangle

$$D_{\mathcal{Y}}\to R\Gamma(\overline{\mathcal{Y}}_{et},\mathbb{Z})\to R\Gamma_W(\overline{\mathcal{Y}},\mathbb{Z})\to D_{\mathcal{Y}}[1]$$

we obtain an exact sequence of abelian groups:

$$\mathrm{Hom}_{\mathcal{D}}(D_{\mathcal{Y}}[1], \mathrm{R}\Gamma_W(\overline{\mathcal{X}}, \mathbb{Z})) \to \mathrm{Hom}_{\mathcal{D}}(R\Gamma_W(\overline{\mathcal{Y}}, \mathbb{Z}), \mathrm{R}\Gamma_W(\overline{\mathcal{X}}, \mathbb{Z}))$$

$$\to \mathrm{Hom}_{\mathcal{D}}(R\Gamma(\overline{\mathcal{Y}}_{et}, \mathbb{Z}), \mathrm{R}\Gamma_W(\overline{\mathcal{X}}, \mathbb{Z}))$$

On the one hand, $\mathrm{Hom}_{\mathcal{D}}(D_{\mathcal{Y}}[1], \mathrm{R}\Gamma_W(\overline{\mathcal{X}}, \mathbb{Z}))$ is uniquely divisible since $D_{\mathcal{Y}}[1]$ is a complex of $\mathbb{Q}$-vector spaces. On the other hand, the abelian group $\mathrm{Hom}_{\mathcal{D}}(\mathrm{R}\Gamma_W(\overline{\mathcal{Y}}, \mathbb{Z}), \mathrm{R}\Gamma_W(\overline{\mathcal{X}}, \mathbb{Z}))$ is finitely generated as it follows from the spectral sequence

$$\prod_{i\in\mathbb{Z}} \mathrm{Ext}^p(H^i_W(\overline{\mathcal{Y}}, \mathbb{Z}), H^{q+i}_W(\overline{\mathcal{X}}, \mathbb{Z})) \Rightarrow H^{p+q}(\mathrm{RHom}(\mathrm{R}\Gamma_W(\overline{\mathcal{X}}, \mathbb{Z}), \mathrm{R}\Gamma_W(\overline{\mathcal{X}}, \mathbb{Z})))$$

since $\mathrm{R}\Gamma_W(\overline{\mathcal{X}}, \mathbb{Z})$ and $\mathrm{R}\Gamma_W(\overline{\mathcal{Y}}, \mathbb{Z})$ are both perfect. Hence the morphism

$$\mathrm{Hom}_{\mathcal{D}}(R\Gamma_W(\overline{\mathcal{Y}}, \mathbb{Z}), \mathrm{R}\Gamma_W(\overline{\mathcal{X}}, \mathbb{Z})) \to \mathrm{Hom}_{\mathcal{D}}(R\Gamma(\overline{\mathcal{Y}}_{et}, \mathbb{Z}), \mathrm{R}\Gamma_W(\overline{\mathcal{X}}, \mathbb{Z}))$$

is injective, which implies the uniqueness of the morphism f^*_W sitting in the commutative square below.

$$\begin{tikzcd}
R\Gamma(\overline{\mathcal{Y}}_{et}, \mathbb{Z}) \arrow[r] \arrow[d, "f^*_{et}"'] & R\Gamma_W(\overline{\mathcal{Y}}, \mathbb{Z}) \arrow[d, "f^*_W"'] \\
R\Gamma(\overline{\mathcal{X}}_{et}, \mathbb{Z}) \arrow[r] & R\Gamma_W(\overline{\mathcal{X}}, \mathbb{Z})
\end{tikzcd}$$

□

Corollary 5.5. *If $\mathcal{X}$ satisfies $\mathbf{L}(\overline{\mathcal{X}}_{et}, d)_{\geq 0}$ then $\mathrm{R}\Gamma_W(\overline{\mathcal{X}}, \mathbb{Z})$ is well defined up to a* unique *isomorphism in $\mathcal{D}$.*

Proof. Let $R\Gamma_W(\overline{\mathcal{X}}, \mathbb{Z})$ and $R\Gamma_W(\overline{\mathcal{X}}, \mathbb{Z})'$ be two complexes as above. By Theorem 5.4, $Id : \mathcal{X} \to \mathcal{X}$ induces a unique isomorphism in $\mathcal{D}$

$$R\Gamma_W(\overline{\mathcal{X}}, \mathbb{Z}) \simeq R\Gamma_W(\overline{\mathcal{X}}, \mathbb{Z})'$$

□

6. Cohomology with finite coefficients

Again $\mathcal{X}$ denotes a proper regular connected arithmetic scheme of dimension d satisfying $\mathbf{L}(\overline{\mathcal{X}}_{et}, d)_{\geq 0}$.

Theorem 6.1. *For any positive integer n the given map between étale and Weil-étale cohomology induces an isomorphism*

$$\mathrm{R}\Gamma_W(\overline{\mathcal{X}}, \mathbb{Z}) \otimes^L_{\mathbb{Z}} \mathbb{Z}/n\mathbb{Z} \simeq \mathrm{R}\Gamma(\overline{\mathcal{X}}_{et}, \mathbb{Z}/n\mathbb{Z})$$

Proof. Consider the exact triangle

$$\mathrm{RHom}(R\Gamma(\mathcal{X}, \mathbb{Q}(d)), \mathbb{Q}[-\delta]) \to \mathrm{R}\Gamma(\overline{\mathcal{X}}_{et}, \mathbb{Z}) \to \mathrm{R}\Gamma_W(\overline{\mathcal{X}}, \mathbb{Z}) \to ...$$

Taking derived tensor product $-\otimes^L_{\mathbb{Z}}\mathbb{Z}/n\mathbb{Z}$ we obtain the following diagram

$$\begin{tikzcd}
\mathrm{RHom}(R\Gamma(\mathcal{X},\mathbb{Q}(d_{\mathcal{X}})),\mathbb{Q}[-\delta]) \arrow[r] \arrow[d, "n"] & \mathrm{R}\Gamma(\overline{\mathcal{X}}_{et},\mathbb{Z}) \arrow[r] \arrow[d, "n"] & \mathrm{R}\Gamma_W(\overline{\mathcal{X}},\mathbb{Z}) \arrow[r] \arrow[d, "n"] & {} \\
\mathrm{RHom}(R\Gamma(\mathcal{X},\mathbb{Q}(d_{\mathcal{X}})),\mathbb{Q}[-\delta]) \arrow[r] \arrow[d] & \mathrm{R}\Gamma(\overline{\mathcal{X}}_{et},\mathbb{Z}) \arrow[r] \arrow[d] & \mathrm{R}\Gamma_W(\overline{\mathcal{X}},\mathbb{Z}) \arrow[r] \arrow[d] & {} \\
0 \arrow[r] & \mathrm{R}\Gamma(\overline{\mathcal{X}}_{et},\mathbb{Z}/n\mathbb{Z}) \arrow[r] & \mathrm{R}\Gamma_W(\overline{\mathcal{X}},\mathbb{Z})\otimes^L_{\mathbb{Z}}\mathbb{Z}/n\mathbb{Z} \arrow[r] & {}
\end{tikzcd}$$

The upper and middle rows are both exact triangles. The colons are all exact as well: for the left colon we simply observe that multiplication by n gives an isomorphism of the complex of $\mathbb{Q}$-vector spaces $\mathrm{RHom}(\mathrm{R}\Gamma(\mathcal{X},\mathbb{Q}(d_{\mathcal{X}})),\mathbb{Q}[-\delta])$; for the central colon we consider the exact sequence of sheaves $0\to\mathbb{Z}\to\mathbb{Z}\to\mathbb{Z}/n\mathbb{Z}\to 0$ on $\overline{\mathcal{X}}_{et}$. It follows that the bottom row is exact, hence the map

$$\mathrm{R}\Gamma(\overline{\mathcal{X}}_{et},\mathbb{Z}/n\mathbb{Z})\simeq\mathrm{R}\Gamma(\overline{\mathcal{X}}_{et},\mathbb{Z})\otimes^L_{\mathbb{Z}}\mathbb{Z}/n\mathbb{Z}\longrightarrow\mathrm{R}\Gamma_W(\overline{\mathcal{X}},\mathbb{Z})\otimes^L_{\mathbb{Z}}\mathbb{Z}/n\mathbb{Z}$$

is an isomorphism in $\mathcal{D}$. □

Corollary 6.2. *For any prime number l and any $i\in\mathbb{Z}$ there is natural isomorphism*

$$H^i_W(\overline{\mathcal{X}},\mathbb{Z})\otimes\mathbb{Z}_l\simeq H^i_{cont}(\overline{\mathcal{X}}_{et},\mathbb{Z}_l)$$

Proof. The previous result yields an exact sequence

$$0\to H^i_W(\overline{\mathcal{X}},\mathbb{Z})_{l^n}\to H^i(\overline{\mathcal{X}}_{et},\mathbb{Z}/l^n\mathbb{Z})\to {}_{l^n}H^{i+1}_W(\overline{\mathcal{X}},\mathbb{Z})\to 0$$

By left exactness of projective limits we get

$$0\to\varprojlim H^i_W(\overline{\mathcal{X}},\mathbb{Z})_{l^n}\to\varprojlim H^i(\overline{\mathcal{X}}_{et},\mathbb{Z}/l^n\mathbb{Z})\to\varprojlim {}_{l^n}H^{i+1}_W(\overline{\mathcal{X}},\mathbb{Z})$$

But $\varprojlim {}_{l^n}H^{i+1}_W(\overline{\mathcal{X}},\mathbb{Z})=0$ since $H^{i+1}_W(\overline{\mathcal{X}},\mathbb{Z})$ is finitely generated. Moreover, we have

$$H^i_{cont}(\overline{\mathcal{X}}_{et},\mathbb{Z}_l)\simeq\varprojlim H^i(\overline{\mathcal{X}}_{et},\mathbb{Z}/l^n\mathbb{Z}).$$

since $H^i(\overline{\mathcal{X}}_{et},\mathbb{Z}/l^n\mathbb{Z})$ is finite.

□

7. Relationship with Lichtenbaum's definition over finite fields

Let Y be a scheme of finite type over a finite field k. We denote by G_k and W_k the Galois group and the Weil group of k respectively. The (small) Weil-étale topos Y^{sm}_W is the category of W_k-equivariant sheaves of sets on the étale site of $\bar{Y}=Y\times_k\bar{k}$. The big Weil-étale topos is defined as the fiber product

$$Y_W:=Y_{et}\times_{\overline{\mathrm{Spec}(\mathbb{Z})}_{et}}\overline{\mathrm{Spec}(\mathbb{Z})}_W\simeq Y_{et}\times_{B^{sm}_{G_k}}B_{W_k}$$

where $B^{sm}_{G_k}$ (resp. B_{W_k}) denotes the small classifying topos of G_k (resp. the big classifying topos of W_k). By [9] one has an exact triangle in the derived category of $Ab(Y_{et})$

$$\mathbb{Z}\to\mathrm{R}\gamma_*\mathbb{Z}\to\mathbb{Q}[-1]\to\mathbb{Z}[1]$$

where $\gamma : Y_W \to Y_{et}$ is the first projection. Applying $\mathrm{R}\Gamma(Y_{et}, -)$ and rotating, we get

$$\mathrm{R}\Gamma(Y_{et}, \mathbb{Q}[-2]) \to \mathrm{R}\Gamma(Y_{et}, \mathbb{Z}) \to \mathrm{R}\Gamma(Y_W, \mathbb{Z}) \to \mathrm{R}\Gamma(Y_{et}, \mathbb{Q}[-2])[1]$$

Proposition 7.1. *Let Y be a d-dimensional connected projective smooth scheme over k satisfying $\mathbf{L}(Y_{et}, d)_{\geq 0}$. Then there is a commutative diagram in $\mathcal{D}$:*

$$\begin{tikzcd}
R\Gamma(Y_{et}, \mathbb{Q}[-2]) \arrow[r] \arrow[d] & R\Gamma(Y_{et}, \mathbb{Z}) \arrow[d, "Id"'] \\
\mathrm{RHom}(\Gamma(Y, \mathbb{Q}(d))_{\geq 0}, \mathbb{Q}[-2d-2]) \arrow[r, "\alpha_Y"] & \mathrm{R}\Gamma(Y_{et}, \mathbb{Z})
\end{tikzcd}$$

where the vertical maps are isomorphisms.

Proof. For simplicity we assume that Y is geometrically connected over k. One has $H^{2d}(Y, \mathbb{Q}(d)) = CH^d(Y)_{\mathbb{Q}} = \mathbb{Q}$ and $H^i(Y, \mathbb{Q}(d)) = 0$ for $i > 2d$. The morphism

$$\mathrm{RHom}_Y(\mathbb{Q}, \mathbb{Q}(d)) \to \mathrm{RHom}(\mathrm{R}\Gamma(Y, \mathbb{Q}), \mathrm{R}\Gamma(Y, \mathbb{Q}(d))) \to \mathrm{RHom}(\mathrm{R}\Gamma(Y, \mathbb{Q}), \mathbb{Q}[-2d])$$

induces a morphism

$$\mathrm{R}\Gamma(Y_{et}, \mathbb{Q}) \simeq \Gamma(Y, \mathbb{Q}) \to \mathrm{RHom}(\Gamma(Y, \mathbb{Q}(d))_{\geq 0}, \mathbb{Q}[-2d]) \tag{13}$$

It follows from Conjecture $\mathbf{L}(Y_{et}, d)_{\geq 0}$, Lemma 4.3 and the fact that $H^i(Y_{et}, \mathbb{Z})$ is finite for $i \neq 0, 2$, that the group $H^i(Y_{et}, \mathbb{Z}(d))_{\geq 0}$ is finite for $i \neq 2d, 2d+2$. This implies that the morphism (13) is a quasi-isomorphism.

It remains to check the commutativity of the diagram. The complex

$$D_Y := \mathrm{RHom}(\Gamma(Y, \mathbb{Q}(d))_{\geq 0}, \mathbb{Q}[-2d-2]) \simeq \mathrm{R}\Gamma(Y_{et}, \mathbb{Q}[-2])$$

is concentrated in degree 2, in particular acyclic in degrees > 2. It follows that the horizontal maps in the diagram of the proposition uniquely factor through the truncated complex $R\Gamma(\overline{Y}_{et}, \mathbb{Z})_{\leq 2}$. It is therefore enough to show that the square

$$\begin{tikzcd}
\mathrm{R}\Gamma(Y_{et}, \mathbb{Q}[-2]) \arrow[r] \arrow[d] & \mathrm{R}\Gamma(Y_{et}, \mathbb{Z})_{\leq 2} \arrow[d, "Id"'] \\
\mathrm{RHom}(\Gamma(Y, \mathbb{Q}(d))_{\geq 0}, \mathbb{Q}[-2d-2]) \arrow[r, "\alpha_Y"] & R\Gamma(Y_{et}, \mathbb{Z})_{\leq 2}
\end{tikzcd}$$

commutes. The exact triangle

$$\mathbb{Z}[0] \to \mathrm{R}\Gamma(Y_{et}, \mathbb{Z})_{\leq 2} \to \pi_1(Y_{et})^D[-2] \to \mathbb{Z}[1]$$

induces an exact sequence of abelian groups

$$\mathrm{Hom}_{\mathcal{D}}(D_Y, \mathbb{Z}[0]) \to \mathrm{Hom}_{\mathcal{D}}(D_Y, \tau_{\leq 2}\mathrm{R}\Gamma(Y_{et}, \mathbb{Z})) \to \mathrm{Hom}_{\mathcal{D}}(D_Y, \pi_1(Y_{et})^D[-2])$$

which shows that the map

$$\mathrm{Hom}_{\mathcal{D}}(D_Y, \tau_{\leq 2}\mathrm{R}\Gamma(Y_{et}, \mathbb{Z})) \to \mathrm{Hom}_{\mathcal{D}}(D_Y, \pi_1(Y_{et})^D[-2])$$

is injective. Indeed, $\mathbb{Z}[0] \simeq [\mathbb{Q} \to \mathbb{Q}/\mathbb{Z}]$ has an injective resolution acyclic in degrees ≥ 2, and D_Y is concentrated in degree 2. In view of the quasi-isomorphism

$$D_Y \simeq \mathrm{Hom}(H^{2d}(Y, \mathbb{Q}(d)), \mathbb{Q})[-2] \simeq H^0(Y, \mathbb{Q})[-2]$$

one is reduced to show the commutativity of the following square (of abelian groups)

$$\begin{tikzcd} H^0(Y_{et},\mathbb{Q}) \arrow[r, "d_2^{0,1}"] \arrow[d] & H^2(Y_{et},\mathbb{Z}) \arrow[d, "Id"'] \\ \mathrm{Hom}(H^{2d}(Y,\mathbb{Q}(d)),\mathbb{Q}) \arrow[r, "H^2(\alpha_Y)"] & H^2(Y_{et},\mathbb{Z}) \end{tikzcd}$$

By construction the map $H^2(\alpha_Y)$ is the following composition

$$H^2(\alpha_Y):\mathrm{Hom}(H^{2d}(Y,\mathbb{Q}(d)),\mathbb{Q})\simeq \mathrm{Hom}(CH^d(Y),\mathbb{Q})\rightarrow CH^d(Y)^D \xleftarrow{\sim} \pi_1(Y_{et})^D \simeq H^2(Y_{et},\mathbb{Z})$$

where the isomorphism $CH^d(Y)^D \xleftarrow{\sim} \pi_1(Y_{et})^D$ is the dual of the reciprocity law of class field theory (which yields an injective morphism with dense image $CH^d(Y)\rightarrow \pi_1(Y_{et})^{ab}$). The upper horizontal map is the differential $d_2^{0,1}$ in the spectral sequence

$$H^i(Y_{et},R^j(\gamma_*)\mathbb{Z})\Rightarrow H^{i+j}(Y_W,\mathbb{Z}).$$

There is a canonical isomorphism

$$H^0(Y_{et},R^1(\gamma_*)\mathbb{Z})=\underrightarrow{lim}_{k'/k}\,\mathrm{Hom}(W_{k'},\mathbb{Z})=\mathrm{Hom}(W_k,\mathbb{Q}),$$

k'/k runs over the finite extensions of k, as it follows from the isomorphism of pro-discrete groups

$$\pi_1(Y'_W,p)\simeq \pi_1(Y'_{et},p)\times_{G_k} W_k.$$

which valid for any Y' connected étale over Y. Then the left vertical map in the square above is the map

$$\deg^*:\mathrm{Hom}(W_k,\mathbb{Q})\rightarrow \mathrm{Hom}(CH^d(Y),\mathbb{Q})$$

induced by the degree map

$$\deg: CH^d(Y)\rightarrow \mathbb{Z}.F\simeq W_k$$

and the differential $d_2^{0,1}$ is the following map:

$$\mathrm{Hom}(W_k,\mathbb{Q})=\mathrm{Hom}(\pi_1(Y_W,p),\mathbb{Q})\rightarrow \pi_1(Y_W,p)^D\simeq \pi_1(Y_{et},p)^D$$

One is therefore reduced to observe that the square

$$\begin{tikzcd} \mathrm{Hom}(W_k,\mathbb{Q}) \arrow[r, "d_2^{0,1}"] \arrow[d, "\deg^*"'] & \pi_1(Y_{et})^D \arrow[d, "Id"'] \\ \mathrm{Hom}(CH^d(Y),\mathbb{Q}) \arrow[r, "H^2(\alpha_Y)"] & \pi_1(Y_{et})^D \end{tikzcd}$$

is commutative.

□

Theorem 7.2. *If Y satisfies $\mathbf{L}(Y_{et},d)_{\geq 0}$ then there is a natural isomorphism in $\mathcal{D}$*

$$\mathrm{R}\Gamma(Y_W,\mathbb{Z})\xrightarrow{\sim}\mathrm{R}\Gamma_W(Y,\mathbb{Z})$$

where the left hand side is the cohomology of the Weil-étale topos and the right hand side is the complex defined in this paper.

Proof. This follows immediately from the previous proposition. □

8. Relationship with Lichtenbaum's definition for number rings

In this section we consider a totally imaginary number field F and let $\mathcal{X} = \mathrm{Spec}(\mathcal{O}_F)$. Recall that $\mathbf{L}(\overline{\mathcal{X}}_{et}, \mathbb{Z}(1))$ holds, hence the complex $\mathrm{R}\Gamma_W(\overline{\mathcal{X}}, \mathbb{Z})$ is well defined.

Theorem 8.1. *There is a quasi-isomorphism*

$$\mathrm{R}\Gamma(\overline{\mathcal{X}}_W, \mathbb{Z})_{\leq 3} \xrightarrow{\sim} \mathrm{R}\Gamma_W(\overline{\mathcal{X}}, \mathbb{Z})$$

where the left hand side is the truncation of Lichtenbaum's complex [22] *and the right hand side is the complex defined in this paper.*

Proof. By [24] Theorem 9.5, we have a quasi-isomorphism

$$R\Gamma(\overline{\mathcal{X}}_{et}, R\mathbb{Z}) \xrightarrow{\sim} R\Gamma(\overline{\mathcal{X}}_W, \mathbb{Z})$$

inducing

$$R\Gamma(\overline{\mathcal{X}}_{et}, R_W\mathbb{Z}) \xrightarrow{\sim} R\Gamma(\overline{\mathcal{X}}_W, \mathbb{Z})_{\leq 3}$$

where $R\mathbb{Z}$ is the complex defined in [24] Theorem 8.5 and $R_W\mathbb{Z} := R\mathbb{Z}_{\leq 2}$. We have an exact triangle

$$\mathbb{Z}[0] \to R_W\mathbb{Z} \to R^2_W\mathbb{Z}[-2]$$

in the derived category of étale sheaves on $\overline{\mathcal{X}}$. Rotating and applying $R\Gamma(\overline{\mathcal{X}}_{et}, -)$ we get an exact triangle

$$R\Gamma(\overline{\mathcal{X}}_{et}, R^2_W\mathbb{Z})[-3] \to R\Gamma(\overline{\mathcal{X}}_{et}, \mathbb{Z}) \to R\Gamma(\overline{\mathcal{X}}_W, \mathbb{Z})_{\leq 3}$$

We have canonical quasi-isomorphisms

$$R\Gamma(\overline{\mathcal{X}}_{et}, R^2_W\mathbb{Z})[-3] \simeq \mathrm{Hom}_{\mathbb{Z}}(\mathcal{O}_F^{\times}, \mathbb{Q})[-3] \simeq \mathrm{RHom}(R\Gamma(\mathcal{X}, \mathbb{Q}(1), \mathbb{Q}[-4])$$

It follows that the morphism $R\Gamma(\overline{\mathcal{X}}_{et}, R^2_W\mathbb{Z})[-3] \to R\Gamma(\overline{\mathcal{X}}_{et}, \mathbb{Z})$ in the triangle above is determined by the induced map

$$\mathrm{Hom}_{\mathbb{Z}}(\mathcal{O}_F^{\times}, \mathbb{Q}) = H^0(\overline{\mathcal{X}}_{et}, R^2_W\mathbb{Z}) \to H^3(\overline{\mathcal{X}}_{et}, \mathbb{Z}) = \mathrm{Hom}_{\mathbb{Z}}(\mathcal{O}_F^{\times}, \mathbb{Q}/\mathbb{Z})$$

and so is the morphism $\alpha_{\mathcal{X}}$. In both cases this map is the obvious one. Hence the square

$$\begin{tikzcd}
R\Gamma(\overline{\mathcal{X}}_{et}, R^2_W\mathbb{Z})[-3] \arrow[r] \arrow[d, "\simeq"'] & R\Gamma(\overline{\mathcal{X}}_{et}, \mathbb{Z}) \arrow[d, "Id"'] \\
\mathrm{RHom}(R\Gamma(\mathcal{X}, \mathbb{Q}(1), \mathbb{Q}[-4]) \arrow[r] & R\Gamma(\overline{\mathcal{X}}_{et}, \mathbb{Z})
\end{tikzcd}$$

is commutative and the result follows. □

9. Cohomology with compact support

We recall below the definition given in [7] of the Weil-étale topos together with its cohomology with $\tilde{\mathbb{R}}$-coefficients. Then we define Weil-étale cohomology with compact support and $\mathbb{Z}$-coefficients.

9.1. **Cohomology with $\tilde{\mathbb{R}}$-coefficients.** Let $\mathcal{X}$ be any proper regular connected arithmetic scheme. The Weil-étale topos is defined as a 2-fiber product of topoi

$$\overline{\mathcal{X}}_W := \overline{\mathcal{X}}_{et} \times_{\overline{\mathrm{Spec}(\mathbb{Z})}_{et}} \overline{\mathrm{Spec}(\mathbb{Z})}_W$$

There is a canonical morphism

$$\mathfrak{f} : \overline{\mathcal{X}}_W \to \overline{\mathrm{Spec}(\mathbb{Z})}_W \to B_{\mathbb{R}}$$

where $B_{\mathbb{R}}$ is Grothendieck's classifying topos of the topological group $\mathbb{R}$ (see [14] or [7]). Consider the sheaf $y\mathbb{R}$ on $B_{\mathbb{R}}$ represented by $\mathbb{R}$ with the standard topology and trivial $\mathbb{R}$-action. Then one defines the sheaf

$$\tilde{\mathbb{R}} := \mathfrak{f}^*(y\mathbb{R})$$

on $\overline{\mathcal{X}}_W$. By [7], the following diagram consists of two pull-back squares of topoi, and the rows give open-closed decompositions:

$$\begin{tikzcd}
\mathcal{X}_W \arrow[r, "\phi"] \arrow[d] & \overline{\mathcal{X}}_W \arrow[d] & \mathcal{X}_{\infty,W} \arrow[l, "i_\infty"'] \arrow[d] \\
\mathcal{X}_{et} \arrow[r, "\varphi"] & \overline{\mathcal{X}}_{et} & Sh(\mathcal{X}_\infty) \arrow[l, "u_\infty"']
\end{tikzcd}$$

Here the map

$$\overline{\mathcal{X}}_W := \overline{\mathcal{X}}_{et} \times_{\overline{\mathrm{Spec}(\mathbb{Z})}_{et}} \overline{\mathrm{Spec}(\mathbb{Z})}_W \longrightarrow \overline{\mathcal{X}}_{et}$$

is the first projection, $Sh(\mathcal{X}_\infty)$ is the category of sheaves on the space $\mathcal{X}_\infty$ and

$$\mathcal{X}_{\infty,W} = B_{\mathbb{R}} \times Sh(\mathcal{X}_\infty)$$

where the product is taken over the final topos $\underline{Sets}$. As shown in [7], the topos $\overline{\mathcal{X}}_W$ has the right $\tilde{\mathbb{R}}$-cohomology with and without compact supports. We have

$$\mathrm{R}\Gamma_W(\overline{\mathcal{X}}, \tilde{\mathbb{R}}) := \mathrm{R}\Gamma(\overline{\mathcal{X}}_W, \tilde{\mathbb{R}}) \simeq \mathrm{R}\Gamma(B_{\mathbb{R}}, \tilde{\mathbb{R}}) \simeq \mathbb{R}[-1] \oplus \mathbb{R}$$

Concerning the cohomology with compact support, one has

$$\mathrm{R}\Gamma_{W,c}(\mathcal{X}, \tilde{\mathbb{R}}) := \mathrm{R}\Gamma(\mathcal{X}_W, \phi_! \tilde{\mathbb{R}}) \simeq \mathrm{R}\Gamma(\mathcal{X}_{et}, \varphi_! \mathbb{R})[-1] \oplus \mathrm{R}\Gamma(\mathcal{X}_{et}, \varphi_! \mathbb{R}) \tag{14}$$

Finally, the complex $\mathrm{R}\Gamma(\mathcal{X}_{et}, \varphi_! \mathbb{R})$ is quasi-isomorphic to

$$\mathrm{Cone}(\mathbb{R}[0] \to \mathrm{R}\Gamma(\mathcal{X}_\infty, \mathbb{R}))[-1].$$

Proposition 9.1. *Cup product with the fundamental class $\theta = Id_{\mathbb{R}} \in H^1(B_{\mathbb{R}}, \mathbb{R})$ yields a morphism*

$$\cup\theta : \mathrm{R}\Gamma_{W,c}(\mathcal{X}, \tilde{\mathbb{R}}) \to \mathrm{R}\Gamma_{W,c}(\mathcal{X}, \tilde{\mathbb{R}})[1]$$

such that the sequence

$$... \to H^{i-1}_{W,c}(\mathcal{X}, \tilde{\mathbb{R}}) \to H^{i}_{W,c}(\mathcal{X}, \tilde{\mathbb{R}}) \to H^{i+1}_{W,c}(\mathcal{X}, \tilde{\mathbb{R}}) \to$$

is an acyclic complex of finite dimensional $\mathbb{R}$-vector spaces.

Proof. In view of $\mathrm{R}\Gamma_{W,c}(\mathcal{X}, \tilde{\mathbb{R}})[1] = \mathrm{R}\Gamma(\mathcal{X}_{et}, \varphi_! \mathbb{R}) \oplus \mathrm{R}\Gamma(\mathcal{X}_{et}, \varphi_! \mathbb{R})[1]$ the map $\cup\theta$ is given by projection and inclusion

$$\mathrm{R}\Gamma_{W,c}(\mathcal{X}, \tilde{\mathbb{R}}) \twoheadrightarrow \mathrm{R}\Gamma(\mathcal{X}_{et}, \varphi_! \mathbb{R}) \hookrightarrow \mathrm{R}\Gamma_{W,c}(\mathcal{X}, \tilde{\mathbb{R}})[1].$$

□

9.2. **Cohomology with $\mathbb{Z}$-coefficients.** In the remaining part of this section, $\mathcal{X}$ denotes a proper regular connected arithmetic scheme of dimension d satisfying $\mathbf{L}(\overline{\mathcal{X}}_{et}, d)_{\geq 0}$. If $\mathcal{X}$ has characteristic p then we set $R\Gamma_{W,c}(\mathcal{X},\mathbb{Z}) := R\Gamma_W(\mathcal{X},\mathbb{Z})$ and the cohomology with compact support is defined as $H^i_{W,c}(\mathcal{X},\mathbb{Z}) := H^i_W(\mathcal{X},\mathbb{Z})$. The case when $\mathcal{X}$ is flat over $\mathbb{Z}$ is the case of interest.

The closed embedding $u_\infty : Sh(\mathcal{X}_\infty) \to \mathcal{X}_{et}$ induces a morphism of complexes

$$u_\infty^* : \mathrm{R}\Gamma(\overline{\mathcal{X}}_{et},\mathbb{Z}) \to \mathrm{R}\Gamma(\mathcal{X}_\infty,\mathbb{Z})$$

where $\mathrm{R}\Gamma(\overline{\mathcal{X}}_{et},\mathbb{Z})$ and $\mathrm{R}\Gamma(\mathcal{X}_\infty,\mathbb{Z})$ are defined using Godement resolutions (which in turn are defined by a conservative system of points of the topos $\overline{\mathcal{X}}_{et}$).

Proposition 9.2. *There exists a unique morphism $i_\infty^* : \mathrm{R}\Gamma_W(\overline{\mathcal{X}},\mathbb{Z}) \to \mathrm{R}\Gamma(\mathcal{X}_{\infty,W},\mathbb{Z})$ which makes the following diagram commutative*

$$\begin{tikzcd}
\mathrm{RHom}(\mathrm{R}\Gamma(\mathcal{X},\mathbb{Q}(d))_{\geq 0},\mathbb{Q}[-\delta]) \arrow[r] \arrow[d] & \mathrm{R}\Gamma(\overline{\mathcal{X}}_{et},\mathbb{Z}) \arrow[r] \arrow[d, "u_\infty^*"] & \mathrm{R}\Gamma_W(\overline{\mathcal{X}},\mathbb{Z}) \arrow[r] \arrow[d, "\exists !"', "i_\infty^*"] & {} \\
0 \arrow[r] & \mathrm{R}\Gamma(\mathcal{X}_\infty,\mathbb{Z}) \arrow[r] & \mathrm{R}\Gamma(\mathcal{X}_{\infty,W},\mathbb{Z}) \arrow[r] & {}
\end{tikzcd}$$

Proof. Recall from [7] that the second projection

$$\mathcal{X}_{\infty,W} := B_{\mathbb{R}} \times Sh(\mathcal{X}_\infty) \to Sh(\mathcal{X}_\infty) = \mathcal{X}_{\infty,et}$$

induces a quasi-isomorphism $\mathrm{R}\Gamma(\mathcal{X}_\infty,\mathbb{Z}) \xrightarrow{\sim} \mathrm{R}\Gamma(\mathcal{X}_{\infty,W},\mathbb{Z})$. Hence the existence of the map i_∞^* will follow (Axiom TR3 of triangulated categories) from the fact that the map

$$\beta : \mathrm{RHom}(\mathrm{R}\Gamma(\mathcal{X},\mathbb{Q}(d))_{\geq 0},\mathbb{Q}[-\delta]) \to \mathrm{R}\Gamma(\overline{\mathcal{X}}_{et},\mathbb{Z}) \to \mathrm{R}\Gamma(\mathcal{X}_\infty,\mathbb{Z}) \tag{15}$$

is the zero map. Again, we set $D_{\mathcal{X}} = \mathrm{RHom}(\mathrm{R}\Gamma(\mathcal{X},\mathbb{Q}(d))_{\geq 0},\mathbb{Q}[-\delta])$ for brevity. Then the uniqueness of i_∞^* follows from the exact sequence

$$\begin{aligned}\mathrm{Hom}_{\mathcal{D}}(D_{\mathcal{X}}[1],\mathrm{R}\Gamma(\mathcal{X}_{\infty,W},\mathbb{Z})) &\xrightarrow{0} \mathrm{Hom}_{\mathcal{D}}(\mathrm{R}\Gamma_W(\overline{\mathcal{X}},\mathbb{Z}),\mathrm{R}\Gamma(\mathcal{X}_{\infty,W},\mathbb{Z})) \\ &\to \mathrm{Hom}_{\mathcal{D}}(\mathrm{R}\Gamma(\overline{\mathcal{X}}_{et},\mathbb{Z}),\mathrm{R}\Gamma(\mathcal{X}_{\infty,W},\mathbb{Z}))\end{aligned}$$

since $\mathrm{Hom}_{\mathcal{D}}(D_{\mathcal{X}}[1],\mathrm{R}\Gamma(\mathcal{X}_{\infty,W},\mathbb{Z}))$ is divisible while $\mathrm{Hom}_{\mathcal{D}}(\mathrm{R}\Gamma_W(\overline{\mathcal{X}},\mathbb{Z}),\mathrm{R}\Gamma(\mathcal{X}_{\infty,W},\mathbb{Z}))$ is finitely generated.

It remains to show that the morphism (15) is indeed trivial. Since $D_{\mathcal{X}}$ is a complex of $\mathbb{Q}$-vector spaces acyclic in degrees < 2, one can choose a (non-canonical) isomorphism

$$D_{\mathcal{X}} \approx \bigoplus_{k\geq 2} H^k(D_{\mathcal{X}})[-k]$$

Then β is identified with the collection of maps $\beta \approx (\beta_k)_{k\geq 2}$ in $\mathcal{D}$, with

$$\beta_k : H^k(D_{\mathcal{X}})[-k] \to \bigoplus_{k\geq 2} H^k(D_{\mathcal{X}})[-k] \approx D_{\mathcal{X}} \to \mathrm{R}\Gamma(\mathcal{X}_\infty,\mathbb{Z})$$

It is enough to show that $\beta_k = 0$ for $k \geq 2$. We fix such a k, and we consider the spectral sequence

$$E_2^{p,q} = \mathrm{Ext}^p(H^k D_{\mathcal{X}}, H^{q+k}(\mathcal{X}_\infty,\mathbb{Z})) \Rightarrow H^{p+q}(\mathrm{RHom}(H^k(D_{\mathcal{X}})[-k],\mathrm{R}\Gamma(\mathcal{X}_\infty,\mathbb{Z})))$$

The group $H^kD_{\mathcal{X}}$ is uniquely divisible and $H^{q+k}(\mathcal{X}_\infty,\mathbb{Z})$ is finitely generated (hence has an injective resolution of length 1), so that $\mathrm{Ext}^p(H^k(D_{\mathcal{X}}),H^{q+k}(\mathcal{X}_\infty,\mathbb{Z}))=0$ for $p\neq 1$. Hence the spectral sequence above degenerates and gives a canonical isomorphism

$$\mathrm{Hom}_{\mathcal{D}}(H^kD_{\mathcal{X}}[-k],\mathrm{R}\Gamma(\mathcal{X}_\infty,\mathbb{Z}))\simeq \mathrm{Ext}^1(H^kD_{\mathcal{X}},H^{k-1}(\mathcal{X}_\infty,\mathbb{Z}))$$

Moreover, the long exact sequence for $\mathrm{Ext}^*(H^kD_{\mathcal{X}},-)$ yields

$$\mathrm{Ext}^1(H^kD_{\mathcal{X}},H^{k-1}(\mathcal{X}_\infty,\mathbb{Z}))\simeq \mathrm{Ext}^1(H^kD_{\mathcal{X}},H^{k-1}(\mathcal{X}_\infty,\mathbb{Z})_{cotor})$$

since the maximal torsion subgroup of $H^{k-1}(\mathcal{X}_\infty,\mathbb{Z})$ is finite and $H^kD_{\mathcal{X}}$ is uniquely divisible. The short exact sequence

$$0\to H^{k-1}(\mathcal{X}_\infty,\mathbb{Z})_{cotor}\to H^{k-1}(\mathcal{X}_\infty,\mathbb{Q})\to H^{k-1}(\mathcal{X}_\infty,\mathbb{Q}/\mathbb{Z})_{div}\to 0$$

is an injective resolution of the $\mathbb{Z}$-module $H^{k-1}(\mathcal{X}_\infty,\mathbb{Z})_{cotor}$. This yields an exact sequence

$$\begin{aligned}0\to \mathrm{Hom}(H^kD_{\mathcal{X}},H^{k-1}(\mathcal{X}_\infty,\mathbb{Q}))\to \mathrm{Hom}(H^kD_{\mathcal{X}},H^{k-1}(\mathcal{X}_\infty,\mathbb{Q}/\mathbb{Z})_{div})\\ \to \mathrm{Ext}^1(H^kD_{\mathcal{X}},H^{k-1}(\mathcal{X}_\infty,\mathbb{Z})_{cotor})\to 0\end{aligned}$$

Let us now define a natural lifting

$$\tilde{\beta}_k\in \mathrm{Hom}(H^kD_{\mathcal{X}},H^{k-1}(\mathcal{X}_\infty,\mathbb{Q}/\mathbb{Z})_{div})$$

of $\beta_k\in\mathrm{Ext}^1(H^kD_{\mathcal{X}},H^{k-1}(\mathcal{X}_\infty,\mathbb{Z})_{cotor})$ and show that this lifting $\tilde{\beta}_k$ is already zero. One can assume $k\geq 2$. Recall that $H^kD_{\mathcal{X}}=\mathrm{Hom}(H^{\delta-k}(\mathcal{X},\mathbb{Q}(d))_{\geq 0},\mathbb{Q})$. We have the following commutative diagram:

$$\begin{tikzcd}
 & \mathrm{Hom}(H^{\delta-k}(\mathcal{X},\mathbb{Q}(d))_{\geq 0},\mathbb{Q}) \arrow[d] \arrow[dr, "H^k(\alpha_k)"] & \\
 & H^{k-1}(\overline{\mathcal{X}}_{et},\mathbb{Q}/\mathbb{Z}) \arrow[r, "\simeq"] \arrow[d] & H^k(\overline{\mathcal{X}}_{et},\mathbb{Z}) \arrow[d] \\
 & H^{k-1}(\mathcal{X}_{\mathbb{Q},et},\mathbb{Q}/\mathbb{Z}) \arrow[r, "\simeq"] \arrow[d] & H^k(\mathcal{X}_{\overline{\mathbb{Q}},et},\mathbb{Z}) \arrow[d] \\
 & H^{k-1}(\mathcal{X}_{\overline{\mathbb{Q}},et},\mathbb{Q}/\mathbb{Z}) \arrow[r, "\simeq"] \arrow[d, "\simeq"] & H^k(\mathcal{X}_{\overline{\mathbb{Q}},et},\mathbb{Z}) \arrow[d] \\
H^{k-1}(\mathcal{X}(\mathbb{C}),\mathbb{Q}) \arrow[r] & H^{k-1}(\mathcal{X}(\mathbb{C}),\mathbb{Q}/\mathbb{Z}) \arrow[r] & H^k(\mathcal{X}(\mathbb{C}),\mathbb{Z})
\end{tikzcd}$$

To make life easier, we assume that $\mathcal{X}(\mathbb{R})=\emptyset$, so that

$$H^{k-1}(\mathcal{X}_\infty,\mathbb{Q}/\mathbb{Z})=H^{k-1}(\mathcal{X}(\mathbb{C}),\mathbb{Q}/\mathbb{Z})^{G_{\mathbb{R}}}$$

The morphism given by the central colon of the diagram above factors through

$$H^{k-1}(\mathcal{X}_\infty,\mathbb{Q}/\mathbb{Z})_{div}\subseteq H^{k-1}(\mathcal{X}_\infty,\mathbb{Q}/\mathbb{Z})\subseteq H^{k-1}(\mathcal{X}(\mathbb{C}),\mathbb{Q}/\mathbb{Z})$$

and yields the desired lifting

$$\tilde{\beta}_k:\mathrm{Hom}(H^{\delta-k}(\mathcal{X},\mathbb{Q}(d)),\mathbb{Q})\to H^{k-1}(\mathcal{X}_\infty,\mathbb{Q}/\mathbb{Z})_{div}$$

The map
$$H^{k-1}(\mathcal{X}_{\mathbb{Q},et}, \mathbb{Q}/\mathbb{Z}) \to H^{k-1}(\mathcal{X}_{\overline{\mathbb{Q}},et}, \mathbb{Q}/\mathbb{Z})$$
factors through $H^{k-1}(\mathcal{X}_{\overline{\mathbb{Q}},et}, \mathbb{Q}/\mathbb{Z})^{G_{\mathbb{Q}}}$ hence so does
$$i \circ \tilde{\beta}_k : \mathrm{Hom}(H^{\delta-k}(\mathcal{X}, \mathbb{Q}(d)), \mathbb{Q}) \to H^{k-1}(\mathcal{X}_\infty, \mathbb{Q}/\mathbb{Z})_{div} \subset H^{k-1}(\mathcal{X}(\mathbb{C}), \mathbb{Q}/\mathbb{Z})$$
where i is the obvious injection. In order to show that $\tilde{\beta}_k = 0$ it is therefore enough to show that
$$(H^{k-1}(\mathcal{X}_{\overline{\mathbb{Q}},et}, \mathbb{Q}/\mathbb{Z})^{G_{\mathbb{Q}}})_{div} = \bigoplus_l (H^{k-1}(\mathcal{X}_{\overline{\mathbb{Q}},et}, \mathbb{Q}_l/\mathbb{Z}_l)^{G_{\mathbb{Q}}})_{div} = 0.$$
Let l be a fixed prime number. Let $U \subseteq Spec(\mathbb{Z})$ on which l is invertible and such that $\mathcal{X}_U \to U$ is smooth. Let $p \in U$. By smooth and proper base change we have:
$$H^{k-1}(\mathcal{X}_{\overline{\mathbb{Q}},et}, \mathbb{Q}_l/\mathbb{Z}_l)^{I_p} \simeq H^{k-1}(\mathcal{X}_{\overline{\mathbb{F}}_p,et}, \mathbb{Q}_l/\mathbb{Z}_l)$$
Recall that $H^{k-1}(\mathcal{X}_{\overline{\mathbb{F}}_p,et}, \mathbb{Z}_l)$ is a finitely generated $\mathbb{Z}_l$-module. We have an exact sequence
$$0 \to H^{k-1}(\mathcal{X}_{\overline{\mathbb{F}}_p,et}, \mathbb{Z}_l)_{cotor} \to H^{k-1}(\mathcal{X}_{\overline{\mathbb{F}}_p,et}, \mathbb{Q}_l) \to H^{k-1}(\mathcal{X}_{\overline{\mathbb{F}}_p,et}, \mathbb{Q}_l/\mathbb{Z}_l)_{div} \to 0$$
We get
$$0 \to (H^{k-1}(\mathcal{X}_{\overline{\mathbb{F}}_p,et}, \mathbb{Z}_l)_{cotor})^{G_{\mathbb{F}_p}} \to H^{k-1}(\mathcal{X}_{\overline{\mathbb{F}}_p,et}, \mathbb{Q}_l)^{G_{\mathbb{F}_p}}$$
$$\to (H^{k-1}(\mathcal{X}_{\overline{\mathbb{F}}_p,et}, \mathbb{Q}_l/\mathbb{Z}_l)_{div})^{G_{\mathbb{F}_p}} \to H^1(G_{\mathbb{F}_p}, H^{k-1}(\mathcal{X}_{\overline{\mathbb{F}}_p,et}, \mathbb{Z}_l)_{cotor})$$
Again, $H^1(G_{\mathbb{F}_p}, H^{k-1}(\mathcal{X}_{\overline{\mathbb{F}}_p,et}, \mathbb{Z}_l)_{cotor})$ is a finitely generated $\mathbb{Z}_l$-module, hence we get a surjective map
$$H^{k-1}(\mathcal{X}_{\overline{\mathbb{F}}_p,et}, \mathbb{Q}_l)^{G_{\mathbb{F}_p}} \to ((H^{k-1}(\mathcal{X}_{\overline{\mathbb{F}}_p,et}, \mathbb{Q}_l/\mathbb{Z}_l)_{div})^{G_{\mathbb{F}_p}})_{div} \to 0$$
Note that
$$((H^{k-1}(\mathcal{X}_{\overline{\mathbb{F}}_p,et}, \mathbb{Q}_l/\mathbb{Z}_l)_{div})^{G_{\mathbb{F}_p}})_{div} = (H^{k-1}(\mathcal{X}_{\overline{\mathbb{F}}_p,et}, \mathbb{Q}_l/\mathbb{Z}_l)^{G_{\mathbb{F}_p}})_{div}$$
By the Weil Conjectures, $H^{k-1}(\mathcal{X}_{\overline{\mathbb{F}}_p,et}, \mathbb{Q}_l)$ is pure of weight $k-1 > 0$. Hence there is no non-trivial element in $H^{k-1}(\mathcal{X}_{\overline{\mathbb{F}}_p,et}, \mathbb{Q}_l)$ fixed by the Frobenius. This shows that
$$(H^{k-1}(\mathcal{X}_{\overline{\mathbb{F}}_p,et}, \mathbb{Q}_l/\mathbb{Z}_l)^{G_{\mathbb{F}_p}})_{div} = H^{k-1}(\mathcal{X}_{\overline{\mathbb{F}}_p,et}, \mathbb{Q}_l)^{G_{\mathbb{F}_p}} = 0$$
hence that
$$(H^{k-1}(\mathcal{X}_{\overline{\mathbb{Q}},et}, \mathbb{Q}_l/\mathbb{Z}_l)^{G_{\mathbb{Q}_p}})_{div} = ((H^{k-1}(\mathcal{X}_{\overline{\mathbb{Q}},et}, \mathbb{Q}_l/\mathbb{Z}_l)^{I_p})^{G_{\mathbb{F}_p}})_{div}$$
$$\simeq (H^{k-1}(\mathcal{X}_{\overline{\mathbb{F}}_p,et}, \mathbb{Q}_l/\mathbb{Z}_l)^{G_{\mathbb{F}_p}})_{div} = 0$$
A fortiori, one has
$$(H^{k-1}(\mathcal{X}_{\overline{\mathbb{Q}},et}, \mathbb{Q}_l/\mathbb{Z}_l)^{G_{\mathbb{Q}}})_{div} = 0$$
and the result follows.

$\square$

Definition 9.3. *There exists an object* $\mathrm{R}\Gamma_{W,c}(\mathcal{X},\mathbb{Z})$*, well defined up to isomorphism in* $\mathcal{D}$*,* endowed with *an exact triangle*

$$\mathrm{R}\Gamma_{W,c}(\mathcal{X},\mathbb{Z}) \to \mathrm{R}\Gamma_W(\overline{\mathcal{X}},\mathbb{Z}) \stackrel{i^*_\infty}{\to} \mathrm{R}\Gamma(\mathcal{X}_{\infty,W},\mathbb{Z}) \to \mathrm{R}\Gamma_{W,c}(\mathcal{X},\mathbb{Z})[1] \tag{16}$$

The determinant $\det_{\mathbb{Z}}\mathrm{R}\Gamma_{W,c}(\mathcal{X},\mathbb{Z})$ *is well defined up to a* canonical *isomorphism.*

The cohomology with compact support is defined (up to isomorphism only) as follows:

$$H^i_{W,c}(\mathcal{X},\mathbb{Z}) := H^i(\mathrm{R}\Gamma_{W,c}(\mathbb{Z}))$$

Remark 9.4. *There should be canonical lifting* $\tilde{\alpha}_{\mathcal{X}} : D_{\mathcal{X}} \to \mathrm{R}\Gamma_c(\mathcal{X}_{et},\mathbb{Z})$ *of the map* $\alpha_{\mathcal{X}}$*. This would make the complex* $\mathrm{R}\Gamma_{W,c}(\mathcal{X},\mathbb{Z})$ *canonically defined.*

To see that $\det_{\mathbb{Z}}\mathrm{R}\Gamma_{W,c}(\mathcal{X},\mathbb{Z})$ is indeed well defined, consider another object $\mathrm{R}\Gamma_{W,c}(\mathcal{X},\mathbb{Z})'$ of $\mathcal{D}$ endowed with an exact triangle (16). There exists a (non-unique) morphism $u : \mathrm{R}\Gamma_{W,c}(\mathcal{X},\mathbb{Z}) \to \mathrm{R}\Gamma_{W,c}(\mathcal{X},\mathbb{Z})'$ lying in a morphism of exact triangles

$$\begin{tikzcd}
R\Gamma(\mathcal{X}_{\infty,W},\mathbb{Z})[-1] \arrow[r] \arrow[d, "Id"'] & R\Gamma_{W,c}(\mathcal{X},\mathbb{Z}) \arrow[r] \arrow[d, "\exists u"', "\simeq"] & R\Gamma_W(\overline{\mathcal{X}},\mathbb{Z}) \arrow[r] \arrow[d, "Id"'] & R\Gamma(\mathcal{X}_{\infty,W},\mathbb{Z}) \arrow[d, "Id"'] \\
R\Gamma(\mathcal{X}_{\infty,W},\mathbb{Z})[-1] \arrow[r] & R\Gamma_{W,c}(\mathcal{X},\mathbb{Z})' \arrow[r] & R\Gamma(\overline{\mathcal{X}},\mathbb{Z}) \arrow[r] & R\Gamma(\mathcal{X}_{\infty,W},\mathbb{Z})
\end{tikzcd}$$

The map u induces

$$\det_{\mathbb{Z}}(u) : \det_{\mathbb{Z}}\mathrm{R}\Gamma_{W,c}(\mathcal{X},\mathbb{Z}) \xrightarrow{\sim} \det_{\mathbb{Z}}\mathrm{R}\Gamma_{W,c}(\mathcal{X},\mathbb{Z})'$$

By [17] p. 43 Corollary 2, $\det_{\mathbb{R}}(u)$ does not depend on the choice of u, since it coincides with the following canonical isomorphism

$$\begin{aligned}
\det_{\mathbb{Z}}\mathrm{R}\Gamma_{W,c}(\mathcal{X},\mathbb{Z}) &\simeq \det_{\mathbb{Z}}\mathrm{R}\Gamma_W(\overline{\mathcal{X}},\mathbb{Z}) \otimes \det^{-1}_{\mathbb{Z}}\mathrm{R}\Gamma(\mathcal{X}_{\infty,W},\mathbb{Z}) \\
&\simeq \det_{\mathbb{Z}}\mathrm{R}\Gamma_{W,c}(\mathcal{X},\mathbb{Z})'
\end{aligned}$$

Given a complex of abelian groups C we write $C_{\mathbb{R}}$ for $C \otimes \mathbb{R}$.

Proposition 9.5. *There is a* canonical and functorial *direct sum decomposition in* $\mathcal{D}$*:*

$$\mathrm{R}\Gamma_W(\overline{\mathcal{X}},\mathbb{Z})_{\mathbb{R}} \simeq \mathrm{R}\Gamma(\overline{\mathcal{X}}_{et},\mathbb{R}) \oplus \mathrm{RHom}(\mathrm{R}\Gamma(\mathcal{X},\mathbb{Q}(d))_{\geq 0},\mathbb{R}[-\delta])[1]$$

such that the following square commutes:

$$\begin{tikzcd}
\mathrm{R}\Gamma_W(\overline{\mathcal{X}},\mathbb{Z})_{\mathbb{R}} \arrow[r, "i^*_\infty \otimes \mathbb{R}"] \arrow[d, "\simeq"'] & \mathrm{R}\Gamma(\mathcal{X}_{\infty,W},\mathbb{Z})_{\mathbb{R}} \arrow[d, "\simeq"'] \\
\mathrm{R}\Gamma(\overline{\mathcal{X}}_{et},\mathbb{R}) \oplus \mathrm{RHom}(\mathrm{R}\Gamma(\mathcal{X},\mathbb{Q}(d))_{\geq 0},\mathbb{R}[-\delta])[1] \arrow[r] & \mathrm{R}\Gamma(\mathcal{X}_\infty,\mathbb{R})
\end{tikzcd}$$

Proof. Applying $(-)\otimes\mathbb{R}$ to the exact triangle of Definition 5.1, we obtain an exact triangle

$$\mathrm{RHom}(\mathrm{R}\Gamma(\mathcal{X},\mathbb{Q}(d))_{\geq 0},\mathbb{R}[-\delta]) \to \mathrm{R}\Gamma(\overline{\mathcal{X}}_{et},\mathbb{R}) \to \mathrm{R}\Gamma_W(\overline{\mathcal{X}},\mathbb{Z})_{\mathbb{R}} \to$$

But the map

$$\mathrm{RHom}(\mathrm{R}\Gamma(\mathcal{X},\mathbb{Q}(d))_{\geq 0},\mathbb{R}[-\delta]) \to \mathrm{R}\Gamma(\overline{\mathcal{X}}_{et},\mathbb{R})$$

is trivial since $\mathrm{R}\Gamma(\overline{\mathcal{X}}_{et},\mathbb{R}) \simeq \mathbb{R}[0]$ is injective and $\mathrm{RHom}(\mathrm{R}\Gamma(\mathcal{X},\mathbb{Q}(d))_{\geq 0},\mathbb{R}[-\delta])$ is acyclic in degrees ≤ 1. This shows the existence of the direct sum decomposition. We write $D_{\mathbb{R}} := \mathrm{RHom}(\mathrm{R}\Gamma(\mathcal{X},\mathbb{Q}(d))_{\geq 0},\mathbb{R}[-\delta])$ and $\mathrm{R}\Gamma(\overline{\mathcal{X}}_{et},\mathbb{R}) \simeq \mathbb{R}[0]$ for brevity. The exact sequence

$$\mathrm{Hom}_{\mathcal{D}}(D_{\mathbb{R}}[1],\mathbb{R}[0]) \to \mathrm{Hom}_{\mathcal{D}}(\mathrm{R}\Gamma_W(\overline{\mathcal{X}},\mathbb{Z}),\mathbb{R}[0])$$

$$\to \mathrm{Hom}_{\mathcal{D}}(\mathbb{R}[0],\mathbb{R}[0]) \to \mathrm{Hom}_{\mathcal{D}}(D_{\mathbb{R}},\mathbb{R}[0])$$

yields an isomorphism $\mathrm{Hom}_{\mathcal{D}}(\mathrm{R}\Gamma_W(\overline{\mathcal{X}},\mathbb{Z}),\mathbb{R}[0]) \xrightarrow{\sim} \mathrm{Hom}_{\mathcal{D}}(\mathbb{R}[0],\mathbb{R}[0])$. Hence there exists a unique map $s_{\overline{\mathcal{X}}} : \mathrm{R}\Gamma_W(\overline{\mathcal{X}},\mathbb{Z})_{\mathbb{R}} \to \mathbb{R}[0]$ such that $s_{\overline{\mathcal{X}}} \circ \gamma^*_{\overline{\mathcal{X}}} = \mathrm{Id}_{\mathbb{R}[0]}$ where $\gamma^*_{\overline{\mathcal{X}}} : \mathbb{R}[0] \to \mathrm{R}\Gamma_W(\overline{\mathcal{X}},\mathbb{Z})$ is the given map. The functorial behavior of $s_{\overline{\mathcal{X}}}$ follows from the fact that it is the unique map such that $s_{\overline{\mathcal{X}}} \circ \gamma^*_{\overline{\mathcal{X}}} = \mathrm{Id}_{\mathbb{R}[0]}$. The direct sum decomposition is therefore canonical and functorial. The commutativity of the square follows from Proposition 9.2. □

Corollary 9.6. *There is a canonical map* $\mathrm{R}\Gamma_W(\overline{\mathcal{X}},\mathbb{Z}) \to \mathrm{R}\Gamma(\overline{\mathcal{X}}_W,\tilde{\mathbb{R}})$.

Proof. For a scheme of characteristic p this map is given by Theorem 7.2. For flat $\mathcal{X}$, it is given by

$$\mathrm{R}\Gamma_W(\overline{\mathcal{X}},\mathbb{Z}) \to \mathrm{R}\Gamma_W(\overline{\mathcal{X}},\mathbb{Z})_{\mathbb{R}} \simeq \mathrm{R}\Gamma(\overline{\mathcal{X}}_{et},\mathbb{R}) \oplus \mathrm{RHom}(\mathrm{R}\Gamma(\mathcal{X},\mathbb{Q}(d))_{\geq 0},\mathbb{R}[-\delta])[1]$$

$$\to \mathrm{R}\Gamma(\overline{\mathcal{X}}_{et},\mathbb{R}) \to \mathrm{R}\Gamma(\overline{\mathcal{X}}_W,\tilde{\mathbb{R}}).$$

□

Proposition 9.7. *There is a (non-canonical) direct sum decomposition*

$$\mathrm{R}\Gamma_{W,c}(\mathcal{X},\mathbb{Z})_{\mathbb{R}} \simeq \mathrm{R}\Gamma_c(\mathcal{X}_{et},\mathbb{R}) \oplus \mathrm{RHom}(\mathrm{R}\Gamma(\mathcal{X},\mathbb{Q}(d))_{\geq 0},\mathbb{R}[-\delta])[1] \tag{17}$$

inducing a canonical *isomorphism*

$$\det_{\mathbb{R}}\mathrm{R}\Gamma_{W,c}(\mathcal{X},\mathbb{Z})_{\mathbb{R}} \simeq \det_{\mathbb{R}}\mathrm{R}\Gamma_c(\mathcal{X}_{et},\mathbb{R}) \otimes \det^{-1}_{\mathbb{R}}\mathrm{RHom}(\mathrm{R}\Gamma(\mathcal{X},\mathbb{Q}(d))_{\geq 0},\mathbb{R}[-\delta])$$

Proof. Again we write $D_{\mathbb{R}} := \mathrm{RHom}(\mathrm{R}\Gamma(\mathcal{X},\mathbb{Q}(d))_{\geq 0},\mathbb{R}[-\delta])$ and $\mathrm{R}\Gamma(\mathcal{X}_{et},\mathbb{R}) \simeq \mathbb{R}[0]$ for brevity (we can assume that $\mathcal{X}$ is connected). Consider the morphism of exact triangles

$$\begin{tikzcd}
R\Gamma_{Wc}(\overline{\mathcal{X}},\mathbb{Z})_{\mathbb{R}} \arrow[r] \arrow[d, "\exists u"'] & R\Gamma_W(\overline{\mathcal{X}},\mathbb{Z})_{\mathbb{R}} \arrow[r, "i^*_\infty\otimes\mathbb{R}"] \arrow[d, "\simeq"'] & R\Gamma(\mathcal{X}_{\infty,W},\mathbb{Z})_{\mathbb{R}} \arrow[r] \arrow[d, "\simeq"'] & {} \\
R\Gamma_c(\overline{\mathcal{X}}_{et},\mathbb{R}) \oplus D_{\mathbb{R}}[1] \arrow[r] & R\Gamma(\overline{\mathcal{X}}_{et},\mathbb{R}) \oplus D_{\mathbb{R}}[1] \arrow[r] & R\Gamma(\mathcal{X}_{\infty},\mathbb{R}) \arrow[r] & {}
\end{tikzcd}$$

Here all the maps but the isomorphism u are canonical. The non-canonical direct sum decomposition (17) follows. A choice of such an isomorphism u induces

$$\det_{\mathbb{R}}(u) : \det_{\mathbb{R}}\mathrm{R}\Gamma_{W,c}(\mathcal{X},\mathbb{Z})_{\mathbb{R}} \xrightarrow{\sim} \det_{\mathbb{R}}(\mathrm{R}\Gamma_c(\mathcal{X}_{et},\mathbb{R}) \oplus D_{\mathbb{R}}[1])$$

But $\det_{\mathbb{R}}(u)$ coincides by [17] with the following (canonical) isomorphism

$$\begin{aligned}\det_{\mathbb{R}}\mathrm{R}\Gamma_{W,c}(\mathcal{X},\mathbb{Z})_{\mathbb{R}} &\simeq \det_{\mathbb{R}}\mathrm{R}\Gamma_{W}(\overline{\mathcal{X}},\mathbb{Z})_{\mathbb{R}}\otimes \det_{\mathbb{R}}^{-1}\mathrm{R}\Gamma(\mathcal{X}_{\infty,W},\mathbb{Z})_{\mathbb{R}}\\ &\simeq \det_{\mathbb{R}}(\mathrm{R}\Gamma(\overline{\mathcal{X}}_{et},\mathbb{R})\oplus D_{\mathbb{R}}[1])\otimes \det_{\mathbb{R}}^{-1}\mathrm{R}\Gamma(\overline{\mathcal{X}}_{et},\mathbb{R})\\ &\simeq \det_{\mathbb{R}}(\mathrm{R}\Gamma_c(\mathcal{X}_{et},\mathbb{R})\oplus D_{\mathbb{R}}[1])\end{aligned}$$

hence does not depend on the choice of u. □

10. The conjecture $\mathbf{B}(\mathcal{X},d)$

Let $\mathcal{X}$ be a proper flat regular connected scheme of dimension d with generic fibre $X=\mathcal{X}_{\mathbb{Q}}$. The "integral part in the motivic cohomology" $H^{2d-1-i}(X_{/\mathbb{Z}},\mathbb{Q}(d))$ is defined as the image of the morphism

$$H^{2d-1-i}(\mathcal{X},\mathbb{Q}(d))\to H^{2d-1-i}(X,\mathbb{Q}(d)).$$

Let $H^p_{\mathcal{D}}(X_{/\mathbb{R}},\mathbb{R}(q))$ denote the real Deligne cohomology and let

$$\rho^i_{\infty}:H^{2d-1-i}_M(X_{/\mathbb{Z}},\mathbb{Q}(d))_{\mathbb{R}}\to H^{2d-1-i}_{\mathcal{D}}(X_{/\mathbb{R}},\mathbb{R}(d))$$

be the Beilinson regulator.

Conjecture 10.1. *(Beilinson) The map ρ^i_{∞} is an isomorphism for $i\geq 1$ and there is an exact sequence*

$$0\to H^{2d-1}_M(X_{/\mathbb{Z}},\mathbb{Q}(d))_{\mathbb{R}}\xrightarrow{\rho^0_{\infty}} H^{2d-1}_{\mathcal{D}}(X_{/\mathbb{R}},\mathbb{R}(d))\to CH^0(X)^*_{\mathbb{R}}\to 0$$

for $i=0$.

Conjecture 10.2. *(Flach) The natural map*

$$H^{2d-1-i}(\mathcal{X},\mathbb{Q}(d))_{\mathbb{R}}\to H^{2d-1-i}(X,\mathbb{Q}(d))_{\mathbb{R}}$$

is injective for $i\geq 0$.

We consider the composite map

$$H^{2d-1-i}(\mathcal{X},\mathbb{Q}(d))_{\mathbb{R}}\to H^{2d-1-i}(X,\mathbb{Q}(d))_{\mathbb{R}}\to H^{2d-1-i}_{\mathcal{D}}(X_{/\mathbb{R}},\mathbb{R}(d)).$$

The conjunction of the two previous conjectures is equivalent to the following

Conjecture 10.3. $\mathbf{B}(\mathcal{X},d)$ *The map*

$$H^{2d-1-i}(\mathcal{X},\mathbb{Q}(d))_{\mathbb{R}}\to H^{2d-1-i}_{\mathcal{D}}(X_{/\mathbb{R}},\mathbb{R}(d))$$

is an isomorphism for $i\geq 1$ and there is an exact sequence

$$0\to H^{2d-1}(\mathcal{X},\mathbb{Q}(d))_{\mathbb{R}}\to H^{2d-1}_{\mathcal{D}}(X_{/\mathbb{R}},\mathbb{R}(d))\to CH^0(X)^*_{\mathbb{R}}\to 0$$

for $i=0$.

If the scheme $\mathcal{X}$ lies over a finite field then Conjecture $\mathbf{B}(\mathcal{X},d)$ is always satisfied by assumption.

11. The regulator map

11.1. **Flat arithmetic schemes.** Let $\mathcal{X}$ be a proper flat regular connected arithmetic scheme of dimension d satisfying $\mathbf{L}(\overline{\mathcal{X}}_{et}, d)$. Let $X = \mathcal{X}_{\mathbb{Q}}$ be the generic fiber. By the work of Goncharov [13], there is a canonical morphism of complexes

$$\Gamma(X, \mathbb{Q}(d)) \to C_{\mathcal{D}}(X_{/\mathbb{R}}, \mathbb{R}(d))$$

inducing Beilinson's regulator on cohomology [4], where the complex $C_{\mathcal{D}}(X_{/\mathbb{R}}, \mathbb{R}(d))$ computes the (truncated) real Deligne cohomology. This complex is truncated so that

$$H^j(C_{\mathcal{D}}(X_{/\mathbb{R}}, \mathbb{R}(d))) = 0 \text{ for } j > 2d - 1.$$

Let $j : X_{Zar} \to \mathcal{X}_{Zar}$ the natural embedding. Consider the map

$$R\Gamma(\mathcal{X}, \mathbb{Q}(d)) \to R\Gamma(\mathcal{X}, j_*\mathbb{Q}(d)) \to R\Gamma(\mathcal{X}, Rj_*\mathbb{Q}(d)) \simeq R\Gamma(X, \mathbb{Q}(d)) \simeq \Gamma(X, \mathbb{Q}(d))$$

In fact we have a quasi-isomorphism $j_*\mathbb{Q}(d) \simeq Rj_*\mathbb{Q}(d)$ since the Zariski-hypercohomology of $\mathbb{Q}(d)$ coincides with the cohomology (of the complex of abelian groups given its global sections) over a field. We consider the composite map

$$\rho_\infty : R\Gamma(\mathcal{X}, \mathbb{Q}(d)) \to \Gamma(X, \mathbb{Q}(d)) \to C_{\mathcal{D}}(X_{/\mathbb{R}}, \mathbb{R}(d))$$

Theorem 11.1. *Assume that Conjecture $\mathbf{B}(\mathcal{X}, d)$ holds. A choice of a direct sum decomposition (17) induces, in a canonical way, an isomorphism in $\mathcal{D}$:*

$$R\Gamma_{W,c}(\mathcal{X}, \mathbb{Z}) \otimes \mathbb{R} \xrightarrow{\sim} R\Gamma_{W,c}(\mathcal{X}, \widetilde{\mathbb{R}})$$

Proof. Duality for Deligne cohomology (see [3] Cor. 2.28)

$$H^i_{\mathcal{D}}(X_{/\mathbb{R}}, \mathbb{R}(p)) \times H^{2d-1-i}_{\mathcal{D}}(X_{/\mathbb{R}}, \mathbb{R}(d-p)) \to \mathbb{R}$$

yields an isomorphism in $\mathcal{D}$

$$C_{\mathcal{D}}(X_{/\mathbb{R}}, \mathbb{R}) \to \mathrm{RHom}(C_{\mathcal{D}}(X_{/\mathbb{R}}, \mathbb{R}(d)), \mathbb{R}[-2d+1])$$

Composing with the quasi-isomorphism

$$R\Gamma(\mathcal{X}_\infty, \mathbb{R}) \to C_{\mathcal{D}}(X_{/\mathbb{R}}, \mathbb{R})$$

we obtain

$$R\Gamma(\mathcal{X}_\infty, \mathbb{R}) \to \mathrm{RHom}(C_{\mathcal{D}}(X_{/\mathbb{R}}, \mathbb{R}(d)), \mathbb{R}[-2d+1])$$

One has a morphism of complexes $C_{\mathcal{D}}(X_{/\mathbb{R}}, \mathbb{R}(d)) \to CH^0(X)^*_{\mathbb{R}}[-2d+1]$ and we define $\tilde{C}_{\mathcal{D}}(X_{/\mathbb{R}}, \mathbb{R}(d))$ to be its Kernel. Applying the functor $\mathrm{RHom}(-, \mathbb{R}[-2d+1])$, we obtain an exact triangle

$$\mathbb{R}[0] \to \mathrm{RHom}(C_{\mathcal{D}}(X_{/\mathbb{R}}, \mathbb{R}(d)), \mathbb{R}[-2d+1]) \to \mathrm{RHom}(\tilde{C}_{\mathcal{D}}(X_{/\mathbb{R}}, \mathbb{R}(d)), \mathbb{R}[-2d+1]) \to \mathbb{R}[1]$$

We have a morphism of exact triangles

$$\begin{tikzcd}
\mathbb{R}[0] \arrow[r] \arrow[d] & R\Gamma(\mathcal{X}_\infty, \mathbb{R}) \arrow[r] \arrow[d] & R\Gamma_c(\mathcal{X}_{et}, \mathbb{R})[1] \arrow[d, "\exists!"'] \\
\mathbb{R}[0] \arrow[r] & \mathrm{RHom}(C_{\mathcal{D}}(X_{/\mathbb{R}}, \mathbb{R}(d)), \mathbb{R}[-2d+1]) \arrow[r] & \mathrm{RHom}(\tilde{C}_{\mathcal{D}}(X_{/\mathbb{R}}, \mathbb{R}(d)), \mathbb{R}[-2d+1])
\end{tikzcd}$$

where the vertical map on the right hand side is uniquely determined. This yields a canonical quasi-isomorphism

$$\mathrm{R}\Gamma_c(\mathcal{X}_{et},\mathbb{R})[-1]\xrightarrow{\sim}\mathrm{RHom}(\tilde{C}_{\mathcal{D}}(X_{/\mathbb{R}},\mathbb{R}(d)),\mathbb{R}[-2d-1]) \tag{18}$$

It follows from Conjecture $\mathbf{B}(\mathcal{X},d)$ that the morphism of complexes

$$\rho_\infty:\mathrm{R}\Gamma(\mathcal{X},\mathbb{Q}(d))\to\Gamma(X,\mathbb{Q}(d))\to C_{\mathcal{D}}(X_{/\mathbb{R}},\mathbb{R}(d))$$

induces a quasi-isomorphism

$$\tilde{\rho}_{\infty,\mathbb{R}}:\mathrm{R}\Gamma(\mathcal{X},\mathbb{Q}(d))_{\mathbb{R}}\xrightarrow{\sim}\widetilde{C}_{\mathcal{D}}(X_{/\mathbb{R}},\mathbb{R}(d)).$$

Applying the functor $\mathrm{RHom}(-,\mathbb{R}[-2d-1])$ and composing with the map (18), we obtain an isomorphism in $\mathcal{D}$:

$$\mathrm{R}\Gamma_c(\mathcal{X}_{et},\mathbb{R})[-1]\xrightarrow{\sim}\mathrm{RHom}(\tilde{C}_{\mathcal{D}}(X_{/\mathbb{R}},\mathbb{R}(d)),\mathbb{R}[-2d-1])\xrightarrow{\sim}\mathrm{RHom}(\mathrm{R}\Gamma(\mathcal{X},\mathbb{Q}(d)),\mathbb{R}[-2d-1]) \tag{19}$$

The inverse isomorphism in $\mathcal{D}$

$$\mathrm{RHom}(\mathrm{R}\Gamma(\mathcal{X},\mathbb{Q}(d)),\mathbb{R}[-2d-1])\xrightarrow{\sim}\mathrm{R}\Gamma_c(\mathcal{X}_{et},\mathbb{R})[-1]$$

and the natural quasi-isomorphism

$$\mathrm{R}\Gamma_c(\mathcal{X}_{et},\mathbb{Z})_{\mathbb{R}}\xrightarrow{\sim}\mathrm{R}\Gamma_c(\mathcal{X}_{et},\mathbb{R})$$

provide us with the desired map in $\mathcal{D}$:

$$\begin{aligned}\mathrm{R}\Gamma_{Wc}(\mathcal{X},\mathbb{Z})_{\mathbb{R}}&\xrightarrow{\sim}\mathrm{R}\Gamma_c(\mathcal{X}_{et},\mathbb{Z})_{\mathbb{R}}\oplus\mathrm{RHom}(\mathrm{R}\Gamma(\mathcal{X},\mathbb{Q}(d)),\mathbb{R}[-2d-1])\\&\xrightarrow{\sim}\mathrm{R}\Gamma_c(\mathcal{X}_{et},\mathbb{R})\oplus\mathrm{R}\Gamma_c(\mathcal{X}_{et},\mathbb{R})[-1]\xrightarrow{\sim}\mathrm{R}\Gamma_{Wc}(\mathcal{X},\tilde{\mathbb{R}})\end{aligned}$$

□

Corollary 11.2. *If Conjecture* $\mathbf{B}(\mathcal{X},d)$ *holds then there is a* canonical *isomorphism*

$$\det_{\mathbb{R}}\mathrm{R}\Gamma_{W,c}(\mathcal{X},\mathbb{Z})_{\mathbb{R}}\xrightarrow{\sim}\det_{\mathbb{R}}\mathrm{R}\Gamma_{W,c}(\mathcal{X},\tilde{\mathbb{R}})$$

Proof. This follows from Proposition 9.7 and Theorem 11.1. □

11.2. **Schemes over finite fields.** Let $\mathcal{X}$ be a proper smooth connected scheme of dimension d over a finite field satisfying $\mathbf{L}(\mathcal{X}_{et},d)$. Here we have $\mathrm{R}\Gamma_{Wc}(\mathcal{X},\mathbb{Z})=\mathrm{R}\Gamma_W(\mathcal{X},\mathbb{Z})$ and a canonical isomorphism

$$\mathrm{R}\Gamma_W(\mathcal{X},\mathbb{Z})_{\mathbb{R}}\simeq\mathrm{R}\Gamma(\mathcal{X}_{et},\mathbb{R})\oplus\mathrm{RHom}(\mathrm{R}\Gamma(\mathcal{X},\mathbb{Q}(d)),\mathbb{R}[-2d-1])$$

It follows from $\mathbf{L}(\mathcal{X}_{et},d)$ that the natural map

$$\mathrm{R}\Gamma(\mathcal{X}_{et},\mathbb{R})[-1]\to\mathrm{RHom}(\mathrm{R}\Gamma(\mathcal{X},\mathbb{Q}(d)),\mathbb{R}[-2d-1])$$

is a quasi-isomorphism (see the proof of Theorem 7.2). This yields a canonical map

$$\begin{aligned}\mathrm{R}\Gamma_W(\mathcal{X},\mathbb{Z})_{\mathbb{R}}&\xrightarrow{\sim}\mathrm{R}\Gamma(\mathcal{X}_{et},\mathbb{R})\oplus\mathrm{RHom}(\mathrm{R}\Gamma(\mathcal{X},\mathbb{Q}(d)),\mathbb{R}[-2d-1])\\&\xrightarrow{\sim}\mathrm{R}\Gamma(\mathcal{X}_{et},\mathbb{R})\oplus\mathrm{R}\Gamma(\mathcal{X}_{et},\mathbb{R})[-1]\xrightarrow{\sim}\mathrm{R}\Gamma_W(\mathcal{X},\tilde{\mathbb{R}})\end{aligned}$$

We obtain the morphism

$$\mathrm{R}\Gamma_W(\mathcal{X},\mathbb{Z})\to\mathrm{R}\Gamma_W(\mathcal{X},\tilde{\mathbb{R}}) \tag{20}$$

inducing the canonical isomorphism

$$\det_{\mathbb{R}}\mathrm{R}\Gamma_W(\mathcal{X},\mathbb{Z})_{\mathbb{R}} \xrightarrow{\sim} \det_{\mathbb{R}}\mathrm{R}\Gamma_W(\mathcal{X},\tilde{\mathbb{R}}) \tag{21}$$

By construction, (20) coincides with

$$\mathrm{R}\Gamma_W(\mathcal{X},\mathbb{Z}) \xleftarrow{\sim} \mathrm{R}\Gamma(\mathcal{X}_W,\mathbb{Z}) \to \mathrm{R}\Gamma(\mathcal{X}_W,\tilde{\mathbb{R}})$$

where the right hand side map is induced by the morphism of sheaves $\mathbb{Z}\to\tilde{\mathbb{R}}$ on the Weil-étale topos $\mathcal{X}_W$.

12. Zeta functions at $s=0$

The following theorem summarizes some results obtained previously.

Theorem 12.1. *Let $\mathcal{X}$ be a proper regular arithmetic scheme.*

a) *The compact support cohomology groups $H^i_{W,c}(\mathcal{X},\tilde{\mathbb{R}})$ are finite dimensional vector spaces over $\mathbb{R}$, vanish for almost all i and satisfy*

$$\sum_{i\in\mathbb{Z}}(-1)^i\dim_{\mathbb{R}} H^i_{W,c}(\mathcal{X},\tilde{\mathbb{R}})=0.$$

b) *Cup product with the fundamental class $\theta\in H^1(B_{\mathbb{R}},\tilde{\mathbb{R}})$ yields an acyclic complex*

$$\cdots\xrightarrow{\cup\theta} H^i_{W,c}(\mathcal{X},\tilde{\mathbb{R}})\xrightarrow{\cup\theta} H^{i+1}_{W,c}(\mathcal{X},\tilde{\mathbb{R}})\xrightarrow{\cup\theta}\cdots$$

c) *If $\mathbf{L}(\overline{\mathcal{X}}_{et},d)_{\geq 0}$ holds, then the compact support cohomology groups $H^i_{W,c}(\mathcal{X},\mathbb{Z})$ are finitely generated over $\mathbb{Z}$ and they vanish for almost all i.*

d) *If $\mathbf{B}(\mathcal{X},d)$ holds then there are isomorphisms*

$$H^i_{W,c}(\mathcal{X},\mathbb{Z})\otimes_{\mathbb{Z}}\mathbb{R}\xrightarrow{\sim} H^i_{W,c}(\mathcal{X},\tilde{\mathbb{R}}).$$

We refer to [17] for generalities on the determinant functor. By Corollary 11.2 we have canonical isomorphisms

$$\begin{aligned}(\det_{\mathbb{Z}}\mathrm{R}\Gamma_{W,c}(\mathcal{X},\mathbb{Z}))_{\mathbb{R}} &\simeq \det_{\mathbb{R}}(\mathrm{R}\Gamma_{W,c}(\mathcal{X},\mathbb{Z})_{\mathbb{R}})\\ &\simeq \det_{\mathbb{R}}\mathrm{R}\Gamma_{W,c}(\mathcal{X},\tilde{\mathbb{R}})\end{aligned}$$

Moreover, the exact sequence b) above provides us with

$$\lambda:\mathbb{R}\xrightarrow{\sim}\bigotimes_{i\in\mathbb{Z}}\det_{\mathbb{R}} H^i_{W,c}(\mathcal{X},\tilde{\mathbb{R}})^{(-1)^i}\xrightarrow{\sim}\det_{\mathbb{R}}\mathrm{R}\Gamma_{W,c}(\mathcal{X},\tilde{\mathbb{R}})\xrightarrow{\sim}(\det_{\mathbb{Z}}\mathrm{R}\Gamma_{W,c}(\mathcal{X},\mathbb{Z}))_{\mathbb{R}}$$

The isomorphism λ can also be defined as follows:

$$\lambda:\mathbb{R}\xrightarrow{\sim}\det_{\mathbb{R}}\mathrm{R}\Gamma_c(\mathcal{X}_{et},\mathbb{R})\otimes\det^{-1}_{\mathbb{R}}\mathrm{R}\Gamma_c(\mathcal{X}_{et},\mathbb{R})\xrightarrow{\sim}$$

$$\det_{\mathbb{R}}(\mathrm{R}\Gamma_c(\mathcal{X}_{et},\mathbb{R})\oplus\mathrm{R}\Gamma_c(\mathcal{X}_{et},\mathbb{R})[-1])\xrightarrow{\sim}\det_{\mathbb{R}}\mathrm{R}\Gamma_{W,c}(\mathcal{X},\tilde{\mathbb{R}})\xrightarrow{\sim}(\det_{\mathbb{Z}}\mathrm{R}\Gamma_{W,c}(\mathcal{X},\mathbb{Z}))_{\mathbb{R}}$$

We also consider the (well defined) integer

$$\mathrm{rank}_{\mathbb{Z}}\mathrm{R}\Gamma_{W,c}(\mathcal{X},\mathbb{Z}):=\sum_{i\in\mathbb{Z}}(-1)^i\cdot i\cdot\mathrm{rank}_{\mathbb{Z}} H^i_{W,c}(\mathcal{X},\mathbb{Z})$$

and we denote by $\zeta^*(\mathcal{X},0)$ the leading Taylor-coefficient of $\zeta(\mathcal{X},s)$ at $s=0$.

Conjecture 12.2. *Let $\mathcal{X}$ be a proper regular connected arithmetic scheme.*

e) *Conjecture* $\mathbf{L}(\overline{\mathcal{X}}_{et}, d)_{\geq 0}$ *holds, the function* $\zeta(\mathcal{X}, s)$ *has a meromorphic continuation to* $s = 0$ *and one has*

$$\mathrm{ord}_{s=0}\zeta(\mathcal{X}, s) = \mathrm{rank}_{\mathbb{Z}}\mathrm{R}\Gamma_{W,c}(\mathcal{X}, \mathbb{Z})$$

f) *Conjecture* $\mathbf{B}(\mathcal{X}, d)$ *holds and one has*

$$\mathbb{Z} \cdot \lambda(\zeta^*(\mathcal{X}, 0)) = \det_{\mathbb{Z}}\mathrm{R}\Gamma_{W,c}(\mathcal{X}, \mathbb{Z})$$

where

$$\lambda : \mathbb{R} \xrightarrow{\sim} \det_{\mathbb{Z}}\mathrm{R}\Gamma_{W,c}(\mathcal{X}, \mathbb{Z}) \otimes \mathbb{R}$$

is the isomorphism defined by cup product with the fundamental class.

13. Relation to Soulé's Conjecture

We fix a regular connected arithmetic scheme $\mathcal{X}$ of dimension d. We assume that $\mathcal{X}$ is flat and projective over $\mathrm{Spec}(\mathbb{Z})$. The following conjecture is a reformulation (see [15] Remark 43) of Soulé's Conjecture [26] in terms of motivic cohomology (thanks to [19] Theorem 14.7 (5)). It presupposes the meromorphic continuation of $\zeta(\mathcal{X}, s)$ at $s = 0$ and the fact that the $\mathbb{Q}$-vector space $H^i(\mathcal{X}, \mathbb{Q}(d))$ is finite dimensional and zero for almost all i.

Conjecture 13.1. *(Soulé) One has*

$$\mathrm{ord}_{s=0}\zeta(\mathcal{X}, s) = \sum_i (-1)^{i+1}\dim_{\mathbb{Q}} H^{2d-i}(\mathcal{X}, \mathbb{Q}(d))$$

Lemma 13.2. *If* $\mathcal{X}$ *satisfies* $\mathbf{L}(\overline{\mathcal{X}}_{et}, d)$ *and* $\mathbf{B}(\mathcal{X}, d)$ *then*

$$\mathrm{rank}_{\mathbb{Z}}\mathrm{R}\Gamma_{W,c}(\mathcal{X}, \mathbb{Z}) = \sum_{i\in\mathbb{Z}} (-1)^i\dim_{\mathbb{Q}} H^{2d+1-i}(\mathcal{X}, \mathbb{Q}(d))$$

Proof. Assuming $\mathbf{L}(\overline{\mathcal{X}}_{et}, d)$ and $\mathbf{B}(\mathcal{X}, d)$ one has

$$\begin{aligned}
\mathrm{rank}_{\mathbb{Z}}\mathrm{R}\Gamma_{W,c}(\mathcal{X}, \mathbb{Z}) &= \sum_{i\in\mathbb{Z}} (-1)^i \cdot i \cdot \dim_{\mathbb{Q}} H^i_{W,c}(\mathcal{X}, \mathbb{Z})_{\mathbb{Q}} \\
&= \sum_{i\in\mathbb{Z}} (-1)^i \cdot i \cdot (\dim_{\mathbb{Q}} H^i_c(\mathcal{X}_{et}, \mathbb{Q}) + \dim_{\mathbb{Q}} H^{2d+2-i-1}(\mathcal{X}, \mathbb{Q}(d))^*) \\
&= \sum_{i\in\mathbb{Z}} (-1)^i \cdot i \cdot (\dim_{\mathbb{Q}} H^{2d-i}(\mathcal{X}, \mathbb{Q}(d))^* + \dim_{\mathbb{Q}} H^{2d+1-i}(\mathcal{X}, \mathbb{Q}(d))^*) \\
&= \sum_{i\in\mathbb{Z}} (-1)^i\dim_{\mathbb{Q}} H^{2d+1-i}(\mathcal{X}, \mathbb{Q}(d))
\end{aligned}$$

The second equality follows from Proposition 9.7. The third equality follows from Conjecture $\mathbf{B}(\mathcal{X}, d)$ and from duality for Deligne cohomology, see the proof of Theorem 11.1. The fourth equality also follows from $\mathbf{B}(\mathcal{X}, d)$ since it implies that $H^{2d-i}(\mathcal{X}, \mathbb{Q}(d))$ is finite dimensional and zero for almost all i. □

Theorem 13.3. *Assume that* $\mathcal{X}$ *satisfies* $\mathbf{L}(\overline{\mathcal{X}}_{et}, d)$ *and* $\mathbf{B}(\mathcal{X}, d)$*. Then Conjecture 12.2 e) is equivalent to Conjecture 13.1.*

Proof. This follows from the previous Lemma. □

14. Relation to the Tamagawa Number Conjecture

Let $\mathcal{O}_F$ be a number ring and let $\mathcal{X} \to \mathrm{Spec}(\mathcal{O}_F)$ be a flat, proper and smooth morphism. Assume that $\mathcal{X}$ is connected of dimension d and satisfies $\mathbf{L}(\overline{\mathcal{X}}_{et}, d)$. Let $Z \subsetneq \mathrm{Spec}(\mathcal{O}_F)$ be a closed subset with open complement U. If $\mathcal{X}_{\mathfrak{p}} := \mathcal{X} \otimes_{\mathcal{O}_F} k(\mathfrak{p})$ satisfies $\mathbf{L}(\mathcal{X}_{\mathfrak{p},et}, d-1)$ for any $\mathfrak{p} \in Z$, then we define $\mathrm{R}\Gamma_{W,c}(\mathcal{X}_U, \mathbb{Z})$ such that the triangle

$$\mathrm{R}\Gamma_{W,c}(\mathcal{X}_U, \mathbb{Z}) \to \mathrm{R}\Gamma_W(\overline{\mathcal{X}}, \mathbb{Z}) \to \bigoplus_{\mathfrak{p} \in Z \cup Z_\infty} \mathrm{R}\Gamma_W(\mathcal{X}_{\mathfrak{p}}, \mathbb{Z}) \tag{22}$$

is exact, where Z_∞ denotes the set of archimedean primes of F. We don't show nor use that $\mathrm{R}\Gamma_{W,c}(\mathcal{X}_U, \mathbb{Z})$ only depends on $\mathcal{X}_U$. In fact $\mathrm{R}\Gamma_{W,c}(\mathcal{X}_U, \mathbb{Z})$ is only defined up to a non-canonical isomorphism but $\det_{\mathbb{Z}}\mathrm{R}\Gamma_{W,c}(\mathcal{X}_U, \mathbb{Z})$ is canonically defined and we have a canonical isomorphism

$$\det{}_{\mathbb{Z}}\mathrm{R}\Gamma_{W,c}(\mathcal{X}_U, \mathbb{Z}) \simeq \det{}_{\mathbb{Z}}\mathrm{R}\Gamma_{W,c}(\mathcal{X}, \mathbb{Z}) \otimes \det{}_{\mathbb{Z}}^{-1}\mathrm{R}\Gamma_W(\mathcal{X}_Z, \mathbb{Z}) \tag{23}$$

We define

$$\mathrm{R}\Gamma_{W,c}(\mathcal{X}_U, \tilde{\mathbb{R}}) := \mathrm{R}\Gamma(\overline{\mathcal{X}}_W, j_!\tilde{\mathbb{R}})$$

where $j : \mathcal{X}_{U,W} \to \overline{\mathcal{X}}_W$ is the natural open embedding of topoi. The triangle

$$\mathrm{R}\Gamma_{W,c}(\mathcal{X}_U, \tilde{\mathbb{R}}) \to \mathrm{R}\Gamma_W(\overline{\mathcal{X}}, \tilde{\mathbb{R}}) \to \bigoplus_{\mathfrak{p} \in Z \cup Z_\infty} \mathrm{R}\Gamma_W(\mathcal{X}_{\mathfrak{p}}, \tilde{\mathbb{R}})$$

is exact, where the map on the right hand side is induced by the closed embedding of topoi $\coprod_{\mathfrak{p} \in Z \cup Z_\infty} \mathcal{X}_{\mathfrak{p},W} \to \overline{\mathcal{X}}_W$ which is the closed complement of the open embedding $j : \mathcal{X}_{U,W} \to \overline{\mathcal{X}}_W$. We obtain a canonical isomorphism

$$\det{}_{\mathbb{R}}\mathrm{R}\Gamma_{W,c}(\mathcal{X}_U, \tilde{\mathbb{R}}) \simeq \det{}_{\mathbb{R}}\mathrm{R}\Gamma_{W,c}(\mathcal{X}, \tilde{\mathbb{R}}) \otimes \det{}_{\mathbb{R}}^{-1}\mathrm{R}\Gamma_W(\mathcal{X}_Z, \tilde{\mathbb{R}}) \tag{24}$$

By Corollary 11.2, (21), (23) and (24) we have a canonical isomorphism

$$\det{}_{\mathbb{R}}\mathrm{R}\Gamma_{W,c}(\mathcal{X}_U, \mathbb{Z})_{\mathbb{R}} \xrightarrow{\sim} \det{}_{\mathbb{R}}\mathrm{R}\Gamma_{W,c}(\mathcal{X}_U, \tilde{\mathbb{R}})$$

Theorem 14.1. *(Flach-Morin) Let $\mathcal{X}$ be a proper flat regular connected arithmetic scheme of dimension d satisfying $\mathbf{L}(\overline{\mathcal{X}}_{et}, d)$ and $\mathbf{B}(\mathcal{X}, d)$.*

1. *If the Hasse-Weil L-functions $L(h^i(\mathcal{X}_{\mathbb{Q}}), s)$ of all motives $h^i(\mathcal{X}_{\mathbb{Q}})$ satisfy the expected meromorphic continuation and functional equation, then Conjecture 12.2 e) holds for $\mathcal{X}$.*
2. *Assume that $\mathcal{X}$ is moreover smooth over a number ring $\mathcal{O}_F$, that $\mathcal{X}$ satisfies Conjectures 1,3,5 of [7] and that $L(h^i(X), s)$ satisfies the expected meromorphic continuation for all i. Assume also that $\mathcal{X}_{\mathfrak{p}} = \mathcal{X} \otimes_{\mathcal{O}_F} k(\mathfrak{p})$ satisfies Conjecture $\mathbf{L}(\mathcal{X}_{\mathfrak{p},et}, d-1)$ for any finite prime $\mathfrak{p}$ of F. Then the Tamagawa number Conjecture ([7] Conjecture 4) for the motive*

$$h(\mathcal{X}_{\mathbb{Q}}) = \bigoplus_{i=0}^{i=2d-2} h^i(\mathcal{X}_{\mathbb{Q}})[-i]$$

is equivalent to statement f) of Conjecture 12.2 for $\mathcal{X}$.

Proof. In view of Theorem 11.1 the first statement is a reformulation of the main result of [7]. By [7] Proposition 9.2, it is enough to check that our definition of the Weil-étale cohomology groups satisfies assumptions a)-j) except perhaps f) of [7] in order to show the second statement. Assumptions a), c), d) and e) are given by Theorem 12.1.

b) follows from [7] Conjecture 1, Lemma 13.2 and Conjecture 10.2 which is part of $\mathbf{B}(\mathcal{X}, d)$.

f) We need Conjecture 12.2 f) to be satisfied by $\mathcal{X}_{\mathfrak{p}} = \mathcal{X} \otimes_{\mathcal{O}_F} k(\mathfrak{p})$ for any finite prime of F. By Theorem 15.1, this follows from Conjecture $\mathbf{L}(\mathcal{X}_{\mathfrak{p},et}, d-1)$.

g) We need this property to hold only in the following special case. By (22) and Theorem 7.2, for any open subset $U \subseteq \mathrm{Spec}(\mathcal{O}_F)$ with closed complement Z we an have exact triangle

$$\mathrm{R}\Gamma_{W,c}(\mathcal{X}_U, \mathbb{Z}) \to \mathrm{R}\Gamma_{W,c}(\mathcal{X}, \mathbb{Z}) \to \mathrm{R}\Gamma(\mathcal{X}_{Z,W}, \mathbb{Z})$$

inducing the canonical isomorphism (23), where $\mathcal{X}_{Z,W}$ is the Weil-étale topos of $\mathcal{X}_Z$.

h) By Corollary 6.2, the natural map $\mathrm{R}\Gamma(\overline{\mathcal{X}}_{et}, \mathbb{Z}) \to \mathrm{R}\Gamma_W(\overline{\mathcal{X}}, \mathbb{Z})$ induces an isomorphism $H^i_W(\overline{\mathcal{X}}, \mathbb{Z}) \otimes \mathbb{Z}_l \simeq H^i(\overline{\mathcal{X}}_{et}, \mathbb{Z}_l)$ for any $i \in \mathbb{Z}$ and any prime number l. We set $V^i_l = H^i(\mathcal{X}_{\overline{\mathbb{Q}},et}, \mathbb{Q}_l)$. By [7] Proposition 9.1 we obtain an isomorphism for $i \geq 0$

$$r^i_l : H^2_f(\mathbb{Q}, V^i_l) \xrightarrow{\sim} H^{i+2}(\overline{\mathcal{X}}_{et}, \mathbb{Q}_l) \xrightarrow{\sim} H^{i+2}_W(\overline{\mathcal{X}}, \mathbb{Z})_{\mathbb{Q}_l}$$

i) follows from the definitions of $\mathrm{R}\Gamma_{Wc}(\mathcal{X}, \mathbb{Z})$ and $\mathrm{R}\Gamma_{Wc}(\mathcal{X}, \tilde{\mathbb{R}})$. Here we obtain a canonical map

$$r^i_\infty : H^i(\mathcal{X}_\infty, \mathbb{R}) \xrightarrow{\sim} H^{i+2}_W(\overline{\mathcal{X}}, \mathbb{Z})_{\mathbb{R}}$$

which is an isomorphism for $i > 0$. Indeed, the map r^i_∞ is induced by the following canonical map:

$$\mathrm{R}\Gamma(\mathcal{X}_\infty, \mathbb{R})[-2] \to \mathrm{R}\Gamma_c(\mathcal{X}_{et}, \mathbb{R})[-1]$$
$$\to \mathrm{RHom}(\mathrm{R}\Gamma(\mathcal{X}, \mathbb{Q}(d)), \mathbb{R}[-2d-1]) \to \mathrm{R}\Gamma_W(\overline{\mathcal{X}}, \mathbb{Z})_{\mathbb{R}}$$

where the second map is (19) and the third map is given by Proposition 9.5.

j) By Lemma 5.2 and Theorem 11.1 we have isomorphisms

$$\lambda^i : H^{i+2}(\overline{\mathcal{X}}_W, \mathbb{Z})_{\mathbb{Q}} \cong H^{2d-1-i}_M(X_{/\mathbb{Z}}, \mathbb{Q}(d))^*$$

for $i \geq 0$ such that $\lambda^i_{\mathbb{R}} \circ r^i_\infty = (\rho^i_\infty)^*$ and $\lambda^i_{\mathbb{Q}_l} \circ r^i_l = \rho^i_l$ (see [7] section 9.4).

∎

15. Proven cases

15.1. **Varieties over finite fields.** Let $\mathbb{F}_q$ be a finite field and let $A(\mathbb{F}_q)$ be the class of smooth projective varieties over $\mathbb{F}_q$ defined in Section 3.2.

Theorem 15.1. *Let $\mathcal{X}$ be a variety over the finite field $\mathbb{F}_q$. If $\mathcal{X}$ lies in $A(\mathbb{F}_q)$ then Conjecture 12.2 holds for $\mathcal{X}$.*

Proof. The variety $\mathcal{X}$ lies in $A(\mathbb{F}_q)$ hence $\mathbf{L}(\mathcal{X}_{et}, d) \Leftrightarrow \mathbf{L}(\mathcal{X}_W, d)$ holds (see Proposition 3.10). In view of Theorem 7.2 the result follows from [21] Theorem 7.4. □

15.2. **Number rings.**

Theorem 15.2. *If $\mathcal{X} = \mathrm{Spec}(\mathcal{O}_F)$ is the spectrum of a number ring, then Conjecture 12.2 holds for $\mathcal{X}$.*

Proof. We set $\mathcal{X} = \mathrm{Spec}(\mathcal{O}_F)$. By Proposition 5.3 one has

$$H^i_W(\mathcal{X}, \mathbb{Z}) = 0 \text{ for } i < 0 \text{ and } i > 3.$$

By Lemma 5.2 one has $H^0_W(\mathcal{X}, \mathbb{Z}) = \mathbb{Z}$, $H^1_W(\mathcal{X}, \mathbb{Z}) = 0$, an exact sequence

$$0 \to H^2(\overline{\mathcal{X}}_{et}, \mathbb{Z})_{codiv} \to H^2_W(\overline{\mathcal{X}}, \mathbb{Z}) \to \mathrm{Hom}(H^1(\overline{\mathcal{X}}_{et}, \mathbb{Z}(1)), \mathbb{Z}) \to 0 \tag{25}$$

and an isomorphism

$$H^3_W(\overline{\mathcal{X}}, \mathbb{Z}) \simeq H^3_W(\overline{\mathcal{X}}_{et}, \mathbb{Z})_{codiv} = ((\mathcal{O}_F^{\times})^D)_{codiv} = \mu_F^D.$$

since $H^0(\overline{\mathcal{X}}_{et}, \mathbb{Z}(1)) = 0$. The sequence (25) reads as follows

$$0 \to Cl(F)^D \to H^2_W(\overline{\mathcal{X}}, \mathbb{Z}) \to \mathrm{Hom}(\mathcal{O}_F^{\times}, \mathbb{Z}) \to 0$$

where $Cl(F)$ is the class group of F and $\mathcal{O}_F^{\times}$ is the units group. The map

$$H^i_{W,c}(\mathcal{X}, \mathbb{Z}) \otimes \mathbb{R} \xrightarrow{\sim} H^i_{W,c}(\mathcal{X}, \tilde{\mathbb{R}})$$

is trivial for $i \neq 1, 2$, the obvious isomorphism for $i = 1$ and the inverse of the transpose of the classical regulator map

$$H^2_{W,c}(\mathcal{X}, \mathbb{Z}) \otimes \mathbb{R} \simeq \mathrm{Hom}(\mathcal{O}_F^{\times}, \mathbb{R}) \to (\prod_{\mathcal{X}_\infty} \mathbb{R})/\mathbb{R}$$

for $i = 2$. The acyclic complex $(H^*_{W,c}(\mathcal{X}, \tilde{\mathbb{R}}), \cup\theta)$ is reduced to the identity map

$$H^1_{W,c}(\mathcal{X}, \tilde{\mathbb{R}}) = (\prod_{\mathcal{X}_\infty} \mathbb{R})/\mathbb{R} \xrightarrow{Id} (\prod_{\mathcal{X}_\infty} \mathbb{R})/\mathbb{R} = H^2_{W,c}(\mathcal{X}, \tilde{\mathbb{R}})$$

The result now follows from the analytic class number formula

$$\mathrm{ord}_{s=0}\zeta(\mathcal{X}, s) = \sharp\mathcal{X}_\infty - 1 \text{ and } \zeta_F^*(0) = -hR/w.$$

where h (respectively w) is the order of $Cl(F)$ (respectively of μ_F) and R is the regulator of the number field F.

□

15.3. **Projective spaces over number rings.** Let B be a Dedekind ring of characteristic 0 with perfect residue fields, and let $p : \mathcal{X} \to B$ be a connected smooth B-scheme of finite type and of absolute dimension $d = \dim(\mathcal{X})$. Recall that $\mathcal{X}_{et}$ denotes the category of sheaves of sets on the small étale site of $\mathcal{X}$ and $\mathcal{X}_{Zar}$ denotes the category of sheaves of sets on the category of étale $\mathcal{X}$-schemes

endowed with the Zariski topology. Let us consider the commutative square of topoi

$$\begin{tikzcd} \mathcal{X}_{et} \arrow[r, "f"] \arrow[d, "\epsilon"] & B_{et} \arrow[d, "\epsilon"] \\ \mathcal{X}_{Zar} \arrow[r, "p"] & B_{Zar} \end{tikzcd}$$

where ϵ is the canonical embedding.

Lemma 15.3. *Let $\mathcal{X}$ be connected smooth over the Dedekind ring B of characteristic 0 with perfect residue fields. Then the map $\epsilon^* Rp_*\mathbb{Z}(n) \to Rf_*(\epsilon^*\mathbb{Z}(n))$ is a quasi-isomorphism for $n \geq dim(\mathcal{X})$.*

Proof. It is enough to check that the induced map on stalks are quasi-isomorphisms for any geometric point $\beta \to B$ over a closed point of B. On the one hand one has

$$\begin{aligned} (\epsilon^* Rp_*\mathbb{Z}(n))_\beta &= (Rp_*\mathbb{Z}(n))_\beta && (26)\\ &\simeq (p_*\mathbb{Z}(n))_\beta && (27)\\ &= \varinjlim p_*\mathbb{Z}(n)(B') && (28)\\ &= \varinjlim \mathbb{Z}(n)(\mathcal{X}\times_B B') && (29)\\ &\simeq \mathbb{Z}(n)(\mathcal{X}\times_B B^{sh}_\beta) && (30) \end{aligned}$$

Here B' runs over the étale neighborhoods of $\beta \to B$, and $B^{sh}_\beta = \varprojlim B'$ denotes the strict henselization. We denote by $(-)_\beta$ the stalk of both a Zariski and étale sheaf, i.e. $(-)_\beta$ denotes the inverse image of the morphisms $\beta_{et} \to B_{et}$ and $\beta_{Zar} \to B_{Zar}$.

(26) is true since taking stalks commutes with sheafification as it follows from the commutative square of topoi

$$\begin{tikzcd} \underline{Set} = \beta_{et} \arrow[r] \arrow[d, "="] & B_{et} \arrow[d, "\epsilon"] \\ \underline{Set} = \beta_{Zar} \arrow[r] & B_{Zar} \end{tikzcd}$$

(27) is a quasi-isomorphism because $p_*\mathbb{Z}(n) \to Rp_*\mathbb{Z}(n)$ is a quasi-isomorphism (see [8] Corollary 3.3 (b)). Then (30) is true since the cycle complex commutes with inverse limits of schemes with affine transition maps (at least if the schemes are quasi-compact).

One the other hand one has

$$\begin{aligned} (Rf_*\mathbb{Z}(n))_\beta &= \varinjlim Rf_*\mathbb{Z}(n)(B') && (31)\\ &= \varinjlim Rf'_*\mathbb{Z}(n)(B') && (32)\\ &= Rf^{sh}_*\mathbb{Z}(n)(B^{sh}_\beta) && (33)\\ &= R\Gamma_{et}(\mathcal{X}\times_B B^{sh}_\beta, \mathbb{Z}(n)) && (34)\\ &\simeq \mathbb{Z}(n)(\mathcal{X}\times_B B^{sh}_\beta) && (35) \end{aligned}$$

Here we denote by f' and f^{sh} the morphisms of étale topoi induced by $\mathcal{X}' = \mathcal{X} \times_B B' \to B'$ and $\mathcal{X}^{sh} = X \times_B B^{sh}_\beta \to B^{sh}_\beta$ respectively. Then (32) is clear since the base change from B to B' is exact and preserves injectives. Take an injective resolution $\mathbb{Z}(n) \simeq I^\bullet$. Then $\mathbb{Z}(n) \mid_{\mathcal{X}'} \simeq I^\bullet \mid_{X'}$ is still an injective resolution, and $\mathbb{Z}(n) \mid_{\mathcal{X}^{sh}} \simeq I^\bullet \mid_{\mathcal{X}^{sh}}$ is an acyclic resolution, since étale cohomology commutes with inverse limit of schemes (quasi-compact and quasi-separated with affine transition maps), and (33) follows. The identity (34) is given by $R\Gamma\,(B^{sh}_{\beta,et}, -) = \Gamma\,(B^{sh}_{\beta,et}, -)$ since $\Gamma\,(B^{sh}_{\beta,et}, -)$ is exact. Finally, the quasi-isomorphism (35) is given by [11] Theorem 7.1. □

Lemma 15.4. *Let $p : \mathbb{P}^m_{B,Zar} \to B_{Zar}$ be the canonical map. Then one has*

$$Rp_*\mathbb{Z}(n) \simeq \bigoplus_{j=0}^{m} \mathbb{Z}(n-j)[-2j]$$

an isomorphism in the derived category $\mathcal{D}(Ab(B_{Zar}))$.

Proof. Consider an open-closed decomposition

$$j : \mathbb{A}^m_B \to \mathbb{P}^m_B \leftarrow \mathbb{P}^{m-1}_B : i$$

We have an exact triangle

$$i_*\mathbb{Z}(n-1)^{\mathbb{P}^{m-1}_B}[-2] \to \mathbb{Z}(n)^{\mathbb{P}^m_B} \to j_*\mathbb{Z}(n)^{\mathbb{A}^m_B}$$

of complexes of Zariski sheaves on $\mathbb{P}^m_B$ (see [8] Corollary 3.3). Applying Rp_* we get

$$p_*i_*\mathbb{Z}(n-1)^{\mathbb{P}^{m-1}_B}[-2] \to p_*\mathbb{Z}(n)^{\mathbb{P}^m_B} \to p_*j_*\mathbb{Z}(n)^{\mathbb{A}^m_B}$$

where we use $p_*\mathbb{Z}(n) \simeq Rp_*\mathbb{Z}(n)$, $p_*i_*\mathbb{Z}(n-1) \simeq R(p_*i_*)\mathbb{Z}(n-1) \simeq (Rp_*)i_*\mathbb{Z}(n-1)$, and $j_*\mathbb{Z}(n) \simeq Rj_*\mathbb{Z}(n)$ (see [8] Corollary 3.3 and the remark after it). But the map on the right hand side has the following section (see [8] Corollary 3.5)

$$p_*j_*\mathbb{Z}(n)^{\mathbb{A}^m_B} \simeq R(p_*j_*)\mathbb{Z}(n)^{\mathbb{A}^m_B} \simeq \mathbb{Z}(n)^B \to p_*\mathbb{Z}(n)^{\mathbb{P}^m_B}$$

Writing $p^m : \mathbb{P}^m_B \to B$, we obtain

$$p^m_*\mathbb{Z}(n)^{\mathbb{P}^m_B} \simeq \mathbb{Z}(n)^B \oplus p^{m-1}_*\mathbb{Z}(n-1)^{\mathbb{P}^{m-1}_B}[-2]$$

By induction we get $p^m_*\mathbb{Z}(n)^{\mathbb{P}^m_B} \simeq \bigoplus_{j=0}^m \mathbb{Z}(n-j)^B[-2j]$. □

Lemma 15.5. *Let F be a number field and let $n \geq 2$. For any $i \leq n+1$, one has*

$$H^i(\mathrm{Spec}(\mathcal{O}_F)_{et}, \mathbb{Z}(n)) \simeq H^i(\mathrm{Spec}(\mathcal{O}_F), \mathbb{Z}(n))$$

and this group is finitely generated. For any $i > n+1$, one has

$$H^i(\mathrm{Spec}(\mathcal{O}_F)_{et}, \mathbb{Z}(n)) \simeq H^i(F_{et}, \mathbb{Z}(n))$$

and this group is finite killed by 2.

Proof. The first statement is given by the Beilinson-Lichtenbaum conjecture (see [8]). By [11] Corollary 7.2, for $n \geq 2$ we have an exact sequence

$$\to \bigoplus_{\mathfrak{p}} H^{i-2}(\mathbb{F}_{\mathfrak{p},et}, \mathbb{Z}(n-1)) \to H^i(\mathrm{Spec}(\mathcal{O}_F)_{et}, \mathbb{Z}(n)) \to H^i(F_{et}, \mathbb{Z}(n)) \to$$

For $i > n+1 \geq 3$ we have $H^{i-2}(\mathbb{F}_{\mathfrak{p},et}, \mathbb{Z}(n-1)) = 0$ hence an isomorphism

$$H^i(\mathrm{Spec}(\mathcal{O}_F)_{et}, \mathbb{Z}(n)) \xrightarrow{\sim} H^i(F_{et}, \mathbb{Z}(n))$$

But $H^i(F_{et}, \mathbb{Q}(n)) = 0$ for $i > n$ hence we obtain

$$H^i(\mathrm{Spec}(\mathcal{O}_F)_{et}, \mathbb{Z}(n)) \xrightarrow{\sim} H^i(F_{et}, \mathbb{Z}(n)) \xrightarrow{\sim} H^{i-1}(F_{et}, \mathbb{Q}/\mathbb{Z}(n))$$

for $i > n+1$. Moreover, for $i-1 > n \geq 2$ we have

$$H^{i-1}(F_{et}, \mathbb{Q}/\mathbb{Z}(n)) \xrightarrow{\sim} H^{i-1}(F_{et}, \mu^{\otimes n}) \xrightarrow{\sim} \sum_{v|\mathbb{R}} H^{i-1}(G_{\mathbb{R}}, \mu^{\otimes n})$$

where $\mu := \varinjlim \mu_m$ denotes the sheaf of all roots of unity, and v runs over the real places of F. Indeed, the first isomorphism follows from $\mathbb{Z}/m\mathbb{Z}(n) \simeq \mu_m^{\otimes n}$, and the second isomorphism is well known (see [23]).

□

Lemma 15.6. *For any $n \geq 2$ and any $i \in \mathbb{Z}$, the map*

$$H^i(\mathrm{Spec}(\mathcal{O}_F), \mathbb{Q}(n)) \to H^i(\mathrm{Spec}(F), \mathbb{Q}(n))$$

is an isomorphism.

Proof. We have a localization sequence

$$... \to \bigoplus_{\mathfrak{p}} H^{i-2}(\mathrm{Spec}(k(\mathfrak{p})), \mathbb{Q}(n-1)) \to H^i(\mathrm{Spec}(\mathcal{O}_F), \mathbb{Q}(n)) \to H^i(\mathrm{Spec}(F), \mathbb{Q}(n)) \to ...$$

where the sum is taken over the finite primes $\mathfrak{p}$ of F (see [8] Corollary 3.4 and correct the obvious misprint). But the motivic cohomology groups

$$H^j(\mathrm{Spec}(k(\mathfrak{p})), \mathbb{Q}(n-1)) = 0 \text{ for all } j \text{ and } n \geq 2$$

are all trivial. □

Theorem 15.7. *Let $\mathcal{O}_F$ be the ring of integers in a number field F and let $\mathcal{X} = \mathbb{P}^n_{\mathcal{O}_F}$ with $n \geq 0$.*

1. *Conjecture $\mathbf{L}(\overline{\mathcal{X}}_{et}, n+1)$ and $\mathbf{B}(\mathcal{X}, n+1)$ hold.*
2. *Conjecture 12.2 e) holds for $\mathcal{X} = \mathbb{P}^n_{\mathcal{O}_F}$.*
3. *If $F/\mathbb{Q}$ is abelian then Conjecture 12.2 f) hold for $\mathcal{X} = \mathbb{P}^n_{\mathcal{O}_F}$.*

Proof.
1. This follows from Lemmas 15.3, 15.5 and 15.6.
2. This follows from Theorem 14.1 and from the fact that the L-function of the motive $h^i(\mathbb{P}^n_F)$ satisfies the meromorphic continuation and the functional equation for any i. Indeed, $h^i(\mathbb{P}^n_F) = h^0(F)(-i/2)$ for i even with $0 \leq i \leq 2n$ and $h^i(\mathbb{P}^n_F) = 0$ otherwise.
3. In view of Theorem 14.1, this will follow from [7] Conjecture 1, 2, 3, 5, for $\mathcal{X} = \mathbb{P}^n_{\mathcal{O}_F}$ and from the Tamagawa number conjecture for the motive $h^0(F)(j)$ with $j \leq 0$. One also needs the validity of Conjecture $\mathbf{L}(\mathbb{P}^n_{k(\mathfrak{p}),et}, n)$ for any finite prime $\mathfrak{p}$ of F. But Conjecture $\mathbf{L}(\mathbb{P}^n_{k(\mathfrak{p}),et}, n)$ is true by the projective bundle formula. The Tamagawa number conjecture for the motive $h^0(F)(j)$ with $j \leq 0$ is known. Conjectures 1, 2 of [7] for $\mathbb{P}^n_{\mathcal{O}_F}$ boil down to Borel's theorem. Conjecture 3 of [7] for $\mathbb{P}^n_{\mathcal{O}_F}$ is a reformulation of Conjecture $\mathbf{Mot}_l$ for the motives $h^{2n-i}(\mathbb{P}^n_F)(n+1)$ and

$h^i(\mathbb{P}^n_F)$, which are either 0 or of the form $h^0(F)(j)$ for suitable $j \in \mathbb{Z}$. Conjecture 5 of [7] for $\mathbb{P}^n_{\mathcal{O}_F}$ is reduced to the following statement: For any prime number p and $j \leq 0$, the complex $R\Gamma_f(\mathbb{Q}_p, \mathbb{Q}_l(j))$ is semi-simple at 0.

□

15.4. **Some open sub-schemes of projective spaces.** Let $F/\mathbb{Q}$ be an abelian number field, let $\mathcal{X} = \mathbb{P}^n_{\mathcal{O}_F}$ and let $\mathcal{U} = \mathbb{P}^n_{\mathcal{O}_F} - Y$ where $Y = \coprod_{i=1}^{i=s} Y_i$ is the disjoint union in $\mathbb{P}^n_{\mathcal{O}_F}$ of varieties Y_i over finite fields $\mathbb{F}_{q_i}$ lying in $A(\mathbb{F}_{q_i})$. We define $\mathrm{R}\Gamma_{W,c}(\mathcal{U}, \mathbb{Z})$ such that the triangle

$$\mathrm{R}\Gamma_{W,c}(\mathcal{U}, \mathbb{Z}) \to \mathrm{R}\Gamma_W(\overline{\mathcal{X}}, \mathbb{Z}) \to \mathrm{R}\Gamma_W(Y, \mathbb{Z}) \oplus \mathrm{R}\Gamma(\mathcal{X}_{\infty,W}, \mathbb{Z}) \to \mathrm{R}\Gamma_{W,c}(\mathcal{U}, \mathbb{Z})[1]$$

is exact. Again $\mathrm{R}\Gamma_{W,c}(\mathcal{U}, \mathbb{Z})$ is only defined up to a non-canonical isomorphism but $\det_{\mathbb{Z}}\mathrm{R}\Gamma_{W,c}(\mathcal{U}, \mathbb{Z})$ is well defined and given with a canonical isomorphism

$$\det_{\mathbb{Z}}\mathrm{R}\Gamma_{W,c}(\mathcal{U}, \mathbb{Z}) \simeq \det_{\mathbb{Z}}\mathrm{R}\Gamma_{W,c}(\mathcal{X}, \mathbb{Z}) \otimes \det^{-1}_{\mathbb{Z}}\mathrm{R}\Gamma_W(Y, \mathbb{Z})$$

Moreover, Corollary 11.2 and (21) yield a canonical isomorphism

$$\lambda_{\mathcal{U}} : \mathbb{R} \xrightarrow{\sim} \det_{\mathbb{Z}}\mathrm{R}\Gamma_{W,c}(\mathcal{U}, \mathbb{Z}) \otimes \mathbb{R}$$

Corollary 15.8. *The following holds:*

$$\mathrm{ord}_{s=0}\zeta(\mathcal{U}, s) = \mathrm{rank}_{\mathbb{Z}}\mathrm{R}\Gamma_{W,c}(\mathcal{U}, \mathbb{Z})$$

$$\mathbb{Z} \cdot \lambda_{\mathcal{U}}(\zeta^*(\mathcal{U}, 0)) = \det_{\mathbb{Z}}\mathrm{R}\Gamma_{W,c}(\mathcal{U}, \mathbb{Z})$$

In particular $\mathrm{rank}_{\mathbb{Z}}\mathrm{R}\Gamma_{W,c}(\mathcal{U}, \mathbb{Z})$ *and* $\det_{\mathbb{Z}}\mathrm{R}\Gamma_{W,c}(\mathcal{U}, \mathbb{Z})$ *only depend on* $\mathcal{U}$.

Proof. In view of Theorem 15.7 and Theorem 15.1, the result follows from the formulas

$$\zeta(\mathcal{X}, s) = \zeta(\mathcal{U}, s) \cdot \zeta(Y, s)$$

$$\mathrm{rank}_{\mathbb{Z}}\mathrm{R}\Gamma_{W,c}(\mathcal{X}, \mathbb{Z}) = \mathrm{rank}_{\mathbb{Z}}\mathrm{R}\Gamma_{W,c}(\mathcal{U}, \mathbb{Z}) + \mathrm{rank}_{\mathbb{Z}}\mathrm{R}\Gamma_W(Y, \mathbb{Z})$$

$$\det_{\mathbb{Z}}\mathrm{R}\Gamma_{W,c}(\mathcal{X}, \mathbb{Z}) \simeq \det_{\mathbb{Z}}\mathrm{R}\Gamma_{W,c}(\mathcal{U}, \mathbb{Z}) \otimes_{\mathbb{Z}} \det_{\mathbb{Z}}\mathrm{R}\Gamma_W(Y, \mathbb{Z})$$

where the last isomorphism is compatible with $(-) \otimes \mathbb{R}$ and λ. □

15.5. **Cohomology of projective spaces in terms of K-theory.** Let F be a number field with r_1 real primes and r_2 complex primes. We set $\mathcal{X} = \mathbb{P}^n_{\mathcal{O}_F}$. *Neglecting the 2-torsion*, the Weil-étale cohomology of $\overline{\mathcal{X}}$ is given by the following

identifications and exact sequences:

$$\begin{array}{rcl} H^i_W(\overline{\mathcal{X}},\mathbb{Z}) & = & \mathbb{Z} \text{ for } i=0, \\ & = & 0 \text{ for } i=1, \\ 0\rightarrow Cl(F)^D\rightarrow H^2_W(\overline{\mathcal{X}},\mathbb{Z}) & \rightarrow & \mathrm{Hom}(\mathcal{O}_F^{\times},\mathbb{Z})\rightarrow 0 \\ & = & \mu_F^D \text{ for } i=3, \\ 0\rightarrow K_2(\mathcal{O}_F)^D\rightarrow H^4_W(\overline{\mathcal{X}},\mathbb{Z}) & \rightarrow & \mathrm{Hom}(K_3(\mathcal{O}_F),\mathbb{Z})\rightarrow 0 \\ & = & (K_3(\mathcal{O}_F)_{tors})^D \text{ for } i=5, \\ & & ... \\ 0\rightarrow K_{2n}(\mathcal{O}_F)^D\rightarrow H^{2n+2}_W(\overline{\mathcal{X}},\mathbb{Z}) & \rightarrow & \mathrm{Hom}(K_{2n+1}(\mathcal{O}_F),\mathbb{Z})\rightarrow 0 \\ & = & (K_{2n+1}(\mathcal{O}_F)_{tors})^D \text{ for } i=2n+3, \\ & = & 0 \text{ for } i>2n+3. \end{array}$$

The complex

$$...\xrightarrow{\cup\theta} H^i_c(\mathcal{X},\mathbb{Z})\otimes\mathbb{R}\xrightarrow{\cup\theta} H^{i+1}_c(\mathcal{X},\mathbb{Z})\otimes\mathbb{R}\xrightarrow{\cup\theta}...$$

is canonically isomorphic to

$$0\longrightarrow\mathbb{R}^{r_1+r_2}/\mathbb{R}\xrightarrow{\sim}\mathrm{Hom}(\mathcal{O}_F^{\times},\mathbb{R})\xrightarrow{0}\mathbb{R}^{r_2}\xrightarrow{\sim}\mathrm{Hom}(K_3(\mathcal{O}_F),\mathbb{R})$$

$$\xrightarrow{0}\mathbb{R}^{r_1+r_2}\xrightarrow{\sim}\mathrm{Hom}(K_5(\mathcal{O}_F),\mathbb{R})\xrightarrow{0}\mathbb{R}^{r_2}\xrightarrow{\sim}\mathrm{Hom}(K_7(\mathcal{O}_F),\mathbb{R})\xrightarrow{0}...$$

in which the isomorphisms are the dual of the regulator maps.

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